

A Test-Particle Analysis of Plasma Turbulence in Astrophysics

by

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by D. E. Hall

ABSTRACT

An important problem of long standing in astrophysics is that of the natural occurrence of charged particles of extremely high energy. The classical form of the problem is that of the origin of cosmic rays, but to this may now be added observations of particles generated in solar flares, in and near the magnetosphere of the earth, in supernovae, in extragalactic radio sources, and in certain laboratory plasmas. In each of these cases, there is reason to believe that the particle acceleration is connected with plasma turbulence or instability.

To investigate the origin of these high-energy particles, an idealized problem has been studied. Test particles are pictured in a general configuration of steady or slowly changing electromagnetic fields, to which an arbitrary spectrum of small random fluctuations is added to represent weak plasma turbulence. The behavior of the particles is found in terms resembling the quasi-linear theory of plasma disturbances, and advantages of this method over the previously used Fokker-Planck approach are described. The limits of applicability of the theory are stated and explained. The result takes the form of a generalized diffusion equation in the phase space; the "diffusion coefficients" are determined by past-history integrals of the second-order correlation functions of the fluctuations, evaluated for pairs of points lying on "unperturbed orbits."

The particular case considered in detail is that of relativistic particles moving in stochastic fields in an otherwise uniformly magnetized plasma. Limiting cases of acceleration, scattering, and spatial diffusion are obtained and discussed; and the results are shown to reduce to those found by other workers (such as Sturrock, Puri, and Jokipii) in appropriate limits. Transverse cyclotron acceleration by low-frequency waves is proposed as the most significant process of stochastic acceleration because of a "selection rule" which is expected to increase its effectiveness as

the particle energy increases, if there is a "universal spectrum" of plasma turbulence for which amplitude is a decreasing function of mode frequency.

The loss of energy by synchrotron radiation is also important for electrons in several of the situations mentioned. This is considered both alone and in combination with stochastic acceleration and other effects, with special attention being given to the possibilities for generating power-law energy spectra. The relevant properties of quasars are reviewed and preliminary comparison is made between the model calculations and the observational data. It is concluded that stochastic acceleration is important in the generation of high-energy particles, that the present calculation has shown how this may reasonably happen, and that this process deserves further and more detailed consideration. Several suggestions are offered for additional observational and theoretical study.

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Chapter 1

ENERGETIC PARTICLES IN NATURE

I. General Introduction

One of the most interesting aspects of the rapidly growing field of astrophysics is the observation, in a wide variety of sources, of evidence for highly suprathermal particles. By this, we mean that there is present a group of particles having energies much higher (in fact, we are thinking primarily of several or many orders of magnitude higher) than the local thermal energy, and that the number of such particles predicted by a Maxwellian velocity distribution would be quite negligible by comparison. Although there are many distinctions that can be made among these different sources, there seems also to be a common thread connecting them. It has been pointed out more than once¹⁻⁴ that this phenomenon is associated with the occurrence of strong turbulence in these astrophysical plasmas. The association provides an explanation of sorts for the origin of the energetic particles. But this tends to take the unsatisfying form of a statement that the energetic particles have some way of approaching an equipartition of energy with the turbulence, and the present limitations of plasma turbulence theory leave us with little information as to further details. This generalization will not be strikingly altered by the present work, for we have chosen only a certain few facets of the problem for study under simplifying assumptions; and we shall ask at the end whether the results can enhance our understanding of the general problem in some way.

II. Examples

In order to have proper background for this study, we first give a brief descriptive review of the principal observed cases of the energetic-particle phenomenon.

A. Solar Flares^{5,6}

These explosive events in the chromosphere of the sun are closely associated with the emission of certain kinds of radio noise.⁷ The

disturbances known as Type III bursts are believed to be due to the excitation of plasma oscillations by streams of electrons that are shot upward from the flare region with velocities of the order of $1/10$ to $1/2$ that of light.⁸ Type IV bursts seem to be synchrotron radiation, indicating relativistic electrons again. Thus, there are many electrons with many keV, or even a few MeV, of energy where nothing else about the flare is generally thought to indicate any possibility of thermal energies more than a very few eV. (A thermal energy of one eV is equivalent to a temperature $T = 11,600^\circ\text{K}.$)

Some large flares are also observed to generate high-energy protons and heavier particles; they are commonly called "proton flares."^{9,10} These particles travel to the earth and beyond with energies as high as a BeV and may be detected by artificial satellites¹¹ and balloons,¹² polar-cap absorption monitors,¹³ or in the case of the stronger events by a substantial increase (especially at higher latitudes) over the usual general cosmic ray activity as routinely monitored on the ground.¹⁴ (The references given here are merely examples from an extensive literature.) The fate of these particles, as well as of the low-energy end of the galactic cosmic-ray spectrum, is one of the most important problems in the study of the interplanetary medium,¹⁵ and parts of the work that will be presented below have potential application in this field.

There is good reason, both observational¹⁶ and theoretical,¹⁷ to believe that the instability underlying a solar flare causes the chromospheric material to become turbulent and break up into a filamentary structure with a scale of the order of one km. It seems likely that the generation of high-energy particles is a normal property of flares; whether we do or do not observe the arrival of protons on the earth, for example, depends on details such as the size of the flare and the configuration of magnetic field lines around the flare,¹⁸⁻²⁰ which may or may not allow the protons to leave the sun.

B. The Earth's Magnetosphere

As the tenuous stream of hot plasma known as the "solar wind" flows outward at supersonic speed from the sun, it gives rise to a standing shock wave when it encounters the earth's magnetic field. Satellite

experiments have shown^{21,22} that electrons with many keV of energy are generated in this bow shock region, and theories have been proposed to explain this.²³⁻²⁵ Satellites far out in the tail of the magnetosphere have also found "islands" of highly energetic electrons²⁶ about which much remains unknown, but which may have important connections with bow shock acceleration or with radiation-belt and auroral phenomena closer to the earth.

The satellite experiments in which the energetic electrons were discovered also recorded a highly turbulent magnetic field in the region around the bow shock and inward to the magnetopause; this has been involved in the explanations referred to above. Correlation between appearances of energetic electrons and fluctuations of magnetic field strength in the magnetospheric tail has recently been reported.²⁷

C. Supernovae

A major portion of our knowledge of these great stellar explosions comes from the Crab Nebula (M1, NGC 1952, Taurus A). The light of the original explosion arrived on earth in A.D. 1054 from a distance of some six thousand light-years. As presently observed, the ejected matter has spread over a region about a light-year or two in size and consists of many globs and filaments. The larger part of its visible radiation is in a bluish continuum which exhibits strong polarization;^{28,29} it is also among the strongest radio and X-ray sources. Shklovskii's explanation³⁰ that we are observing synchrotron radiation is now widely accepted and, as will be seen below, has been found useful in explaining other unusual objects more recently discovered. According to a recent estimate,³¹ the radiating electrons are in a field of the order of 10^{-3} gauss and have energies in the approximate range 10^8 to 3×10^{12} eV, the differential spectrum $N(E)$ varying as $E^{-1.6}$. [The number of particles with energy between E and $E + dE$ is $N(E) dE$.]

There is no direct evidence to confirm it, but the presence of such highly energetic electrons has made it very attractive to think that there are similar numbers of nuclei present with comparable or even higher energies, and this has made supernovae as a class a leading candidate for the source of the Galactic cosmic rays.^{32,33}

D. Cosmic Rays³²⁻³⁴

In contrast to the other examples mentioned here, extensive research in cosmic rays dates back over fifty years. The interaction of these particles with the earth's atmosphere, however, has taken up a large portion of this work and is not of interest here. When information about the "primary" cosmic ray flux was finally unraveled, it pertained mostly to the following three very remarkable properties:

1. The chemical composition grossly resembles that of the rest of the universe, but upon closer inspection the medium and heavy nuclei are somewhat overabundant with respect to hydrogen while the light nuclei Li, Be and B are overabundant by five or six orders of magnitude. The interpretation of these abundances involves the abundances in the source(s), the possibility of preferential acceleration,³⁵ and the certainty of differing rates of loss by collision during transit from source to observer.
2. The intensity is found to be isotropic and constant in time to high accuracy, implying in some sense either a nonlocal and non-transient source or an efficient scattering mechanism.
3. The energy spectrum extends to extremely high particle energies, at least of the order of 10^{20} eV³⁶ but probably not much more.³⁷ The dependence of number upon energy is drastically nonthermal; except for a slight kink near 10^{15} eV, it is well described by a negative power law with an exponent of about 2.5. This differential energy spectrum is especially interesting in the present context.

Fermi's attempts^{38,39} to explain the acceleration of these particles were not really successful, but have had great importance in stimulating further work on the problem. The current view tends to split the situation into two parts: first is "injection" of energetic particles into the galactic or intergalactic background by some source, or class of sources, and second is the subsequent storage, stirring, diffusion and loss in the ambient medium. The Russian workers in particular have studied this second part in considerable detail. Fermi's continuing acceleration is regarded as too small to be of interest, because the parameters that determine it may be independently observed and they do not have suitable values.⁴⁰ Thus the principal burden of acceleration is thrown back upon the injecting sources. The detailed explanation of the acceleration within the sources in terms of their local properties remains very much unsolved, even though the division explained above enables this to be isolated

from many other aspects of cosmic ray study so that they may be studied and solved independently.

A special difficulty in the study of possible sources of injection is that direct observations give information only about energetic electrons, and all properties proposed for energetic nuclei in the cosmic ray injection accelerator can be compared with these observations only by way of poorly understood indirect inferences. We shall comment further on the possible solution of this difficulty in Chapter 6.

E. Extragalactic Radio Sources

There are a number of elliptical and irregular galaxies in which some sort of explosive event seems to have taken place and transformed them into "radio galaxies."^{41,42} One of the more notable examples is M82 (NGC 3034, 3C231) with its well-known Doppler velocity profile;⁴³ another is M87 (NGC 4486, 3C274, Virgo A) with its strange jet emitting polarized light.⁴⁴ The prototype of repeated explosions, each creating a double radio source, is NGC 5128 (Centaurus A). An example of the group known as Seyfert galaxies is NGC 1275 (3C84, Perseus A). Again we have evidence of energetic electrons responsible for the radiation, but how to connect this with the probable coexistence of energetic protons and nuclei of higher charge is still an outstanding problem.⁴⁵ These radio galaxies are all of significant interest here, and we wish to emphasize the similarities between their properties and those of the more dramatic quasi-stellar objects.^{46,47}

The explanation of the redshifts of the "quasars" is still the subject of extended discussion.⁴⁸⁻⁵² Terrell has been foremost in advancing reasons⁵¹ for associating them with our Galaxy or with other nearby galaxies, mainly in order to reduce the energy requirements; but we are not yet satisfied with the ideas being advanced⁵² for the redshifts in this picture. Although we still prefer the cosmological distance interpretation and shall write in terms of it below, the quasars would still be relevant to this discussion on the "local hypothesis." At least some of the radiation from the compact cores of these objects probably must be explained by a collective mechanism,⁵³ greatly complicating the question of what particle energies may be present. The theory for these objects

in general is still very difficult and uncertain,^{46-48,54-56} but, in the exceptional case of 3C273, we again encounter a most interesting optical jet extending outward some 150,000 light-years⁵⁷ and terminating in a radio cloud.⁵⁸ It may be treated to a large extent independently of the more drastic conditions in the nucleus, and its interpretation in terms of synchrotron radiation seems to require^{57,59} electrons with energies as high as 10^{12} eV in a magnetic field with strength of the order of 10^{-4} gauss. Thus, as in supernovae and radio galaxies, even the most conservative interpretation involves highly energetic electrons and gives strong reason to suspect the coexistence of heavy particles of even higher energy. In particular, the quasi-stellar objects (or perhaps some or all radio galaxies as well⁶⁰) may provide a source for the highest-energy cosmic rays,⁵⁹ which cannot be readily explained in local Galactic terms alone.

F. Laboratory Plasmas

These are on a vastly smaller scale, and are not directly within the immediate purposes of this work. But we mention them here as another example of originally unexpected high-energy particles^{61,62} which were accelerated in a turbulent plasma in the presence of beam-plasma instability. The explanation has been dealt with in a somewhat controversial article by Stix.⁶³ Stochastic cyclotron heating of electrons in a plasma contained in a mirror machine by applied noise fields has also been experimentally investigated,⁶⁴ with results in basic agreement with the accompanying theory by Puri.⁶⁵

III. Plan of Study

There are several things which have seemed to us to be important in understanding how these various acceleration phenomena occur and how they may be related to each other, and which have called for further study.

First, there is the role of turbulence. Rather than take on the whole difficult problem of plasma turbulence,⁶⁶ we have chosen to take the following viewpoint: We feel that turbulence is present and is important somehow in natural particle acceleration; we study what effect any given spectrum of turbulence could have; we try to make intelligent

guesses as to what general sort of spectrum might be reasonable and likely; and we trust and hope that other work will be done which will follow an appropriate plasma instability into its fully nonlinear regime and show more precisely just what spectrum of turbulence should result. We must also leave for separate investigation the effects upon the turbulence of the loss of energy to the accelerated particles.

Second, there is the role of radiation. In all the cases above where relativistic electrons are thought to be present, the principal means of knowing of their existence is to observe their radiation in a magnetic field. The loss of energy through radiation is, in turn, a strong and important influence on the amount of energy the particles may have. This subject has received some study,⁶⁷ but is by no means exhausted.

Third, there is the relationship between the electrons and the heavy particles. This has been the subject of important speculation,^{45,68,69} but we do not have precise answers to several questions: Are high-energy electrons and nuclei always produced together? What are the relative numbers? What differences may there be in the type of energy spectrum? Do the answers to these questions assist us in a clearer understanding of the origin of cosmic rays?

We outline here the plan for the remainder of this work. In Chapters 2 and 3 the turbulence of a plasma is idealized by a spectrum of uncorrelated waves and the associated electric and magnetic fields, which will exhibit certain properties of randomness. We investigate by statistical methods the effect these fields will have on a test particle subject to them, and translate the results into the behavior of a large number of particles subject to the fluctuating fields but uncorrelated with each other. In Chapter 4, we present a brief summary of the effect of synchrotron radiation upon a single particle and upon a distribution of particles with differing energies and other properties. Chapter 5 constitutes the step from abstract theory to hopefully realistic prediction, in which we consider how these processes might work in combination with each other or with still different processes to determine the spectra of particles to be found in nature. Finally, in Chapter 6 we compare these theoretical considerations with some of the properties of the observed phenomena and evaluate the extent to which this investigation has improved our understanding of the problems discussed in this opening chapter.

IV. Preview of Results

In order that the details of the calculations to follow should not obscure the ideas being advanced, we give here a brief qualitative summary of what is to be found. We shall conclude that randomly fluctuating electromagnetic fields may account for the production of high-energy particles by the process of "stochastic acceleration," and that the most important electric field components for this purpose are those transverse to the average magnetic field and varying with a frequency which matches the natural frequency of gyration of the particles being accelerated. Stochastic acceleration is an average effect for a large number of particles, when any one particle may either gain or lose energy in small increments; it is equivalent to a generalized diffusion or "random walk" process in energy.

Whenever there is a net average acceleration of particles there must be a transfer of energy from some source. In Fermi's discussion of cosmic ray acceleration, this source was provided by the interstellar "clouds"; the interaction between particles and clouds was equivalent to thermalizing collisions between two species of particles at different temperatures in a gas, and because of the very great mass of the clouds they had a high effective temperature and provided a source of energy for acceleration of the cosmic ray particles. In this work, the source of energy is taken to be plasma turbulence; in particular, we shall find that low-frequency hydromagnetic turbulence seems most promising, so this is similar to Fermi's description of the energy source in spite of the greater abstraction.

The plasma turbulence, in turn, must have its energy supplied by some form of instability, which may be either strong and catastrophic or weak and continuing.⁷⁰ But this chain will not be traced in detail, for it is the purpose of this work to take advantage of plasma turbulence as an established fact whereby certain natural plasmas have a reservoir of energy upon which the stochastic acceleration process may draw.

Chapter 2

RELATIVISTIC STOCHASTIC ACCELERATION: THE METHOD

I. Introduction

The need for consideration of the behavior of charged particles under the influence of stochastic electromagnetic fields has been indicated in the previous chapter. It has also been discussed briefly by Sturrock in his exposition of the basic method to be presented here. In that paper,⁷¹ hereinafter called "SA," the stochastic acceleration of non-relativistic particles in fluctuating electric fields was studied in the small-gyroradius limit, assuming that the fields were properly described statistically as a stationary random process. It is important, in order to apply this theory to solar flares and other more energetic astronomical events, to extend the consideration of stochastic acceleration to relativistic energies and magnetic fields, which we shall now do. The present chapter will be taken up in developing a general formalism, and in the following chapter we shall consider some more or less tractable special cases. (Parts of these two chapters have been presented in less detail in a recent article.⁷²) Spatial diffusion effects also result from this general theory, but they are not our primary interest here and will be de-emphasized.

More specifically, we shall now present two different approaches to the stochastic acceleration problem, although both lead to the same results. Without implying complete equivalence to other contexts in which the terms are used, we label these as the Fokker-Planck (FP) and quasi-linear (QL) methods. The FP method was used throughout the greater part of our work on this problem, and gives contact with certain previous work,^{71,73} but now tends to be more of historical interest since the QL method has been found to add both simplicity and elegance to the calculations. Nevertheless, it seems useful to discuss both because of the unique insights each can contribute and the differing shades of meaning they attribute to the theory.

The presentation in this chapter will be very general, but probably most clearly understood if it is kept in mind that we intend later to specify that the fluctuating forces arise from electromagnetic fields.

These in turn are to be thought of as aspects of turbulent modes of a plasma, but this work will be limited to the use of the fields themselves, leaving for separate study the problem of relating the electromagnetic spectrum to the spectrum of plasma excitations which it represents. These excitations must be sufficiently small ("weak" turbulence) that the conditions on the fields for a perturbation theory, which will explicitly appear below, can be satisfied. There is another and nonequivalent sense in which these excitations may be small; namely, that they be linear oscillations, so that each Fourier mode in the spectrum is independent and does not interact with any other modes. This latter sense of smallness is not of importance to our theory, for the spectrum function $S(\vec{k}, \omega)$ is still well defined and well behaved when the oscillations are nonlinear (even though it no longer has such a simple intuitive meaning); all the results will remain valid as long as the fluctuations are small in the first sense.

II. Assumptions and Notation

The problem to be treated here is one of test particles; that is, even though many particles are being considered in some sense, we will ignore their effect upon one another as well as their reaction upon whatever source, external to them, may be providing the forces which determine their motion. We may suggest three ways in which such a situation could be of interest. First, this may simply be regarded as treating half of the complete problem, with the other half--the dynamics of the external sources--still to be done. This is the way in which this analysis could be related to quasi-linear theory. Second, it may be proper to consider the forces as undisturbed because they are actually being imposed by a source with very high effective "impedance." Finally, the particles we consider may actually be some special class of particles, many in number but still only a very small fraction of all the particles present, so that the spectrum of forces is determined and supplied by the background and is indeed not much affected by the "test distribution." A group of highly suprathermal particles in a non-Maxwellian "tail" could provide such a case.

The other assumption defining the problem is the idealization of the force fluctuations as a stationary, homogeneous, random field with zero mean, which is completely defined as a statistical process by its correlation functions of all orders. The results will be usable for non-stationary, inhomogeneous field spectra if and when the time or length characteristic of growth or damping is much greater than that characteristic of self-correlation. Since the field fluctuations will be supposed to have a finite correlation time they must, when formally Fourier-analyzed, necessarily have a spectrum consisting of more than one mode of oscillation. There will be finite effective half-widths in frequency and wave-number related to the effective self-correlation time and length by

$$\omega_{1/2} T_C \cong 1, \quad k_{1/2} L_C \cong 1, \quad (2.2.1)$$

and in these terms the conditions for quasi-stationary turbulence are

$$|\omega_i| \ll \omega_{1/2}, \quad |k_i| \ll k_{1/2}, \quad (2.2.2)$$

where the damping rates ω_i and k_i are evaluated at any real frequency or wavenumber which makes a significant contribution to the spectrum.

Now we introduce the notation with which we shall solve this problem. First, let the particles be represented by a distribution function F which gives their density in phase space, and let X_μ ($\mu = 1, 2, \dots, 6$) stand for any six orthogonal coordinates (usually three of position and three of momentum) in this phase space. Then the total number of particles is

$$N = \int \prod_{\nu=1}^6 (h_\nu dX_\nu) F(X_\mu, t); \quad (2.2.3)$$

the h_ν are square roots of the elements of a diagonal metric tensor $[(d\vec{X})_\mu = h_\mu dX_\mu]$, and in Cartesian coordinates $h_\nu = 1$ for all ν . In curvilinear coordinates, the action of the standard vector operator $\vec{\nabla}$

upon a scalar S and a 6-vector \vec{V} is given by

$$\nabla_{\mu} S = (\text{grad } S)_{\mu} = \frac{1}{h_{\mu}} \frac{\partial S}{\partial X_{\mu}}, \quad \nabla_{\mu} V_{\mu} = \text{div } \vec{V} = \frac{1}{h} \frac{\partial}{\partial X_{\mu}} \left(\frac{h V_{\mu}}{h_{\mu}} \right) \quad (2.2.4)$$

with

$$h = \prod_{\nu=1}^6 h_{\nu}.$$

The usual summation convention for repeated subscripts may be used with the understanding that the subscript on h_{ν} is not to be counted.

The equations of motion of a single particle situated at the point X_{μ} at the time t will depend upon the fields of force to which the particle is subject. We shall write these equations as

$$\frac{dX_{\mu}}{dt} = G_{\mu} [X_{\nu}(t), t] + g_{\mu} [X_{\nu}(t), t], \quad (2.2.5)$$

where G_{μ} includes the effects of whatever large-scale determinate field may be present, and g_{μ} is the contribution of the randomly fluctuating fields. Solutions of this equation for $g_{\mu} = 0$ will be referred to as "unperturbed orbits."

Since it is our intention to deal exclusively with electromagnetic forces, we may note here a property which simplifies several equations below. It is well known that the momentum divergence of the Lorentz force, vanishes (as it does for any force, such as gravitational, which is derived from a potential). That is, the 3-vectors \vec{x} and \vec{p} satisfy

$$\frac{d}{d\vec{p}} \cdot \left(\frac{d\vec{p}}{dt} \right) = 0 \quad (2.2.6a)$$

and

$$\frac{d}{d\vec{X}} \cdot \left(\frac{d\vec{X}}{dt} \right) = 0 , \quad (2.2.6b)$$

the latter equality following because \vec{x} and \vec{p} are treated as independent coordinates in phase space. Thus we will have

$$\frac{1}{h} \frac{\partial}{\partial X_\mu} (h g_\mu) = 0 \quad (2.2.7a)$$

and

$$\frac{1}{h} \frac{\partial}{\partial X_\mu} (h g_\mu) = 0 . \quad (2.2.7b)$$

III. Fokker-Planck Approach^{74,75}

There is one basic equation that lies at the foundation of the present theory; this is the equation of continuity (or conservation of particles) in phase space. Although in later chapters we will include at some points the effects of nuclear or other interactions capable of creating or annihilating particles, we write this equation now in the absence of any sources or sinks:

$$\frac{\partial F}{\partial t} + \vec{\nabla} \cdot \left[\frac{d\vec{X}}{dt} F \right] \equiv \frac{\partial F}{\partial t} + \frac{1}{h} \frac{\partial}{\partial X_\mu} \left[h \frac{dX_\mu}{dt} F \right] = 0 . \quad (2.3.1)$$

FP and QL represent different ways of working with this equation to find the effects of a particular type of forces in (2.2.5), that is, different stratagems for integrating (2.3.1) even though the forces acting are not completely known.

FP is best understood as a multidimensional Lagrange expansion; we may begin with equation (4.1) of Reference 75, translated into the present notation:

$$\begin{aligned}
F(X_\sigma, t + \Delta t) = F(X_\sigma, t) - \frac{1}{h} \frac{\partial}{\partial X_\mu} [hF(X_\sigma, t) \Delta X_\mu(X_\sigma, t, \Delta t)] \\
+ \frac{1}{2h} \frac{\partial}{\partial X_\mu} \frac{\partial}{\partial X_\nu} [hF(X_\sigma, t) \Delta X_\mu(X_\sigma, t, \Delta t) \Delta X_\nu(X_\sigma, t, \Delta t)] - \dots
\end{aligned}
\tag{2.3.2}$$

This relates the change in F at the point X_σ during a small time interval Δt to the change in position ΔX_μ of a single particle starting at X_σ during this same interval, and in the limit of infinitesimal Δt it of course reduces to (2.3.1). But for finite Δt the expression

$$\Delta X_\mu(X_\sigma, t, \Delta t) = \int_0^{\Delta t} dt' \frac{dX_\mu}{dt} [X_\sigma + \Delta X_\sigma(X_\sigma, t, t'), t+t']
\tag{2.3.3}$$

must be used.

In order to circumvent the implicit nature of this equation and make actual use of it, we must carry out a Taylor expansion of its integrand. Consider first the case $g_\mu = 0$ and evaluate all quantities at X_σ, t ; then

$$\Delta X_\mu^G = G_\mu \Delta t + \left[\frac{\partial G_\mu}{\partial t} + G_\nu \frac{\partial G_\mu}{\partial X_\nu} \right] \frac{(\Delta t)^2}{2} + \dots
\tag{2.3.4}$$

The interval Δt must at this point be left free to take on any value that may be required in treating g_μ when the latter becomes finite, so the only satisfactory condition for this expansion will be

$$\frac{DG_\mu}{Dt} \equiv \frac{\partial G_\mu}{\partial t} + G_\nu \frac{\partial G_\mu}{\partial X_\nu} \equiv 0 ;
\tag{2.3.5}$$

it will also preclude any contribution to (2.3.4) by terms of third or higher order in Δt . This is a restriction, not on G_μ , but on what

systems of coordinates may be used. For a given G_μ there may well be several systems for which this is satisfied, and there is always at least one (although it is not guaranteed to be familiar, attractive, or convenient). To show this, consider again the particle at X_σ ; this particle may be followed both forward and backward in time as it traces out its orbit in phase space. Likewise, a particle at another nearby point traces out a nearby orbit; no two orbits may ever intersect (for then Newtonian mechanics would be indeterminate), and every point of the space has exactly one orbit passing through it. Then we have only to construct a five-dimensional subspace orthogonal to this family of trajectories. In that five-space, any coordinates at all may be used, for they are only a parametrization, or labeling, of the orbits and the corresponding G_μ are zero. The sixth coordinate may then give distance along the orbits as proportional to the readings of a clock moving with the particles, so that G_6 is nonzero but constant, and satisfies (2.3.5).

There must also be a restriction on F in order that (2.3.2) converge:

$$\Delta t \left| G_\mu \frac{\partial F}{\partial X_\mu} \right| \ll F. \quad (2.3.6)$$

An alternative view is that for a given F the theory will succeed only if all other requirements below can be met with a Δt small enough to satisfy (2.3.6). The meaning of this should be clear; it is that no calculated contribution to the change in F should be comparable to F itself.

Returning to (2.3.3), we calculate the contribution of g_μ by Taylor-expanding about $X'_\sigma(t+t') = X_\sigma + G_\sigma(X_\sigma, t)t'$ rather than about X_σ itself:

$$\Delta X_\mu^g = \int_0^{\Delta t} dt' \left\{ g_\mu(X'_\sigma, t+t') + \frac{\partial g_\mu}{\partial X_\nu} (X'_\sigma, t+t') \int_0^{t'} dt'' [g_\nu(X''_\sigma, t+t'') + \dots] \right\}. \quad (2.3.7)$$

The fields g_μ are not known in detail, and equation (2.3.2) must be averaged in order to obtain an answer in terms of their statistical properties alone. Since the fields have zero mean, (2.3.7) leads to

$$\langle \Delta X_\mu^g \rangle = \int_0^{\Delta t} dt' \int_0^{t'} dt'' \left\langle g_\nu(X''_\sigma, t+t'') \frac{\partial g_\mu}{\partial X_\nu}(X'_\sigma, t+t') \right\rangle + O(g^4) , \quad (2.3.8)$$

$$\langle \Delta X_\mu^g \Delta X_\nu^g \rangle = \int_0^{\Delta t} dt' \int_0^{\Delta t} dt'' \left\langle g_\mu(X'_\sigma, t+t') g_\nu(X''_\sigma, t+t'') \right\rangle + O(g^4) , \quad (2.3.9)$$

$$\langle \Delta X_\mu^g \Delta X_\nu^g \Delta X_\rho^g \rangle = O(g^4) , \quad \text{etc.} \quad (2.3.10)$$

Clearly, a perturbation-type calculation is being made with respect to the strength of the fluctuating fields; we are finding terms of second order to represent the lowest-order nontrivial effect, and are neglecting all higher-order terms. The condition for doing this is also required if (2.3.2) is to converge strongly, and is just like (2.3.6):

$$\Delta t \left| g_\mu \frac{\partial F}{\partial X_\mu} \right| \ll F . \quad (2.3.11)$$

Finally, consider the meaning of (2.3.8) and (2.3.9). They depend on the correlation between the values of g_μ at two different points on the same unperturbed orbit. A useful result can be obtained, independent of the transient details of g_μ , only by requiring that Δt be much larger than the typical self-correlation time T_C of g_μ , as can be seen from Fig. 1; for then one integration can be extended to infinity in both directions with small relative error, and the other simply gives a linear proportionality to Δt . This is precisely the behavior one must have in order to use these quantities as "Fokker-Planck coefficients." The form customarily used is obtained by dividing (2.3.2) by Δt ; if

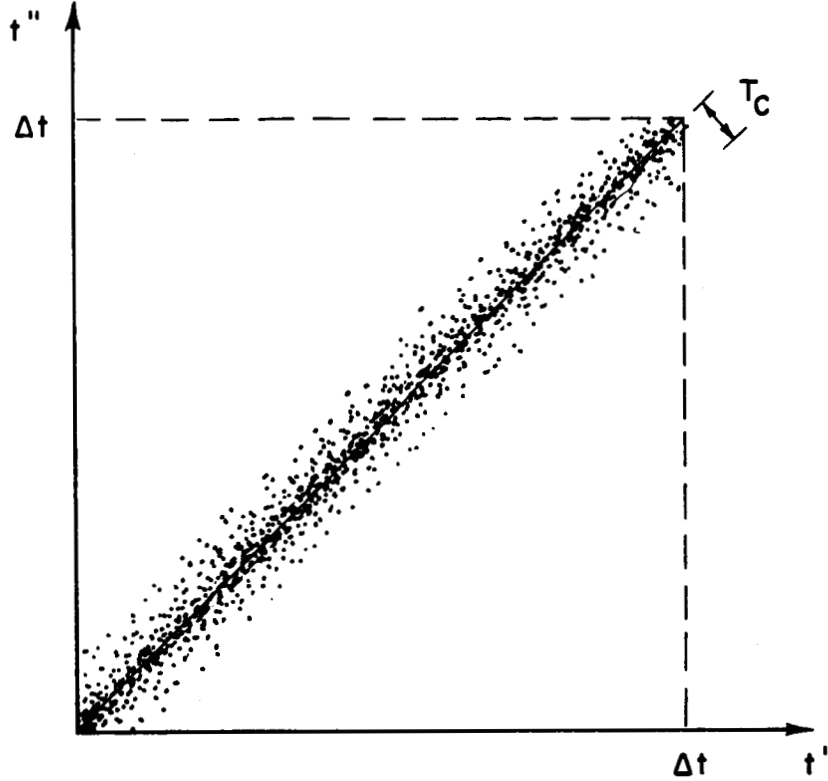


FIG. 1. REGION OF INTEGRATION OF $\langle g_\mu(t') g_\nu(t'') \rangle$.
Outside the shaded area, the integrand becomes very small.

(2.3.6) is satisfied, $\Delta F/\Delta t$ may still be represented by $\partial F/\partial t$, and after using (2.3.4), (2.3.5) and (2.2.7) we have

$$\frac{DF}{Dt} \equiv \frac{\partial F}{\partial t} + G_\mu \frac{\partial F}{\partial X_\mu} \approx -\frac{1}{h} \frac{\partial}{\partial X_\mu} \left[hF \left\langle \frac{\Delta X_\mu}{\Delta t} \right\rangle \right] + \frac{1}{2h} \frac{\partial}{\partial X_\mu} \frac{\partial}{\partial X_\nu} \left[hF \left\langle \frac{\Delta X_\mu \Delta X_\nu}{\Delta t} \right\rangle \right]. \quad (2.3.12)$$

This equation was the basis of our early work on the problem, but it had certain disadvantages. The "diffusion" coefficient (2.3.9) is generally readily calculable, for it is written entirely in terms of quantities on the unperturbed orbit. But the "friction" coefficient (2.3.8) makes use, in effect, of first-order orbit corrections. One consequence of this is simply algebraic complication, which in principle is

not disastrous; but in actual calculation for a special case, this complication was of sufficient quantity that the appearance of the (lengthy) final results did not cause us to realize the presence of an error at an early step, and incorrect conclusions were actually drawn from these results⁷⁶ (as shown in the next chapter, magnetic scattering is isotropic). A second consequence was the appearance of some terms which seemed to have an improper asymptotic dependence on Δt , so that they could not be used as Fokker-Planck coefficients. This apparently limited the possibilities for complete solution to a few special cases (nonrelativistic, no electric field, or small gyrofrequency); but as will be seen by following the QL method in the next section, this was because the algebraic complication camouflaged the way in which the terms combine with one another to eliminate these difficulties.

We must mention still another condition, which is easily overlooked. In order to pass from a single-particle result to a statistical average, it was implicitly assumed that each particle was subject to a different, independent member of an ensemble of field representations. If this is to be true, the average nearest-neighbor distance between test particles should be large compared with the coherence length of the fluctuations. Otherwise the particles would be, in Buneman's picturesque description, "all rocking in the same boat" rather than diffusing independently. We might try to circumvent this by averaging over a larger region of space and so effectively defining a smoothed distribution function which would properly obey our equations. But one should be rather cautious about adopting this attitude, because if there are N particles close enough to each other to be responding coherently to the fluctuations, they can react upon the spectrum with a strength proportional to N^2 . This particular point appears more clearly from FP than from QL.

IV. Quasi-Linear Approach⁷²

We now present a different mathematical technique for analyzing the same problem, which removes some of the difficulties encountered with FP and makes it possible to write concisely a more general solution than formerly seemed possible. This derivation bears a strong similarity to the quasi-linear theory of plasma disturbances, and in particular it

approaches in some respects recent work of Shapiro^{77,78} and of Kennel and Engelmann.⁷⁹ But it differs in the representation of the fields, and in the exact meaning to be attached to these fields and to the distribution function describing the particles.

We begin again with equation (2.3.1), which is exact. The function F will develop in an irregular way under the influence of g_μ , and the detailed fluctuations are not of interest. We need to find only some expectation value of F in terms of the statistical properties of g_μ , so we consider an ensemble of distribution functions, all beginning with identical values at some time $t = t_0$. Let each of these functions be subject to a different member of an ensemble of realizations of g_μ , i.e., fluctuating field histories which are independent of one another in detail, but identical as to statistical averages. At any time $t > t_0$ the various functions F will differ from each other, and an equation is required for $\langle F \rangle$, the average of F over all members of the ensemble. This is provided by taking the average of (2.3.1), using (2.2.5) and (2.2.7):

$$\frac{D\langle F \rangle}{Dt} \equiv \frac{\partial \langle F \rangle}{\partial t} + g_\mu \frac{\partial \langle F \rangle}{\partial X_\mu} = - \frac{1}{h} \frac{\partial}{\partial X_\mu} \left\langle h g_\mu \delta F \right\rangle, \quad (2.4.1)$$

where $\delta F = F - \langle F \rangle$. Then the difference between (2.3.1) and (2.4.1) is

$$\frac{D\delta F}{Dt} = - g_\mu \frac{\partial \langle F \rangle}{\partial X_\mu} - g_\mu \frac{\partial \delta F}{\partial X_\mu} + \left\langle g_\mu \frac{\partial \delta F}{\partial X_\mu} \right\rangle; \quad (2.4.2)$$

the last two equations together contain all the information in (2.3.1). [The use of (2.27) to move the derivative in some terms but not in others is determined by the desire to write the final result in the form (2.4.7).]

We shall again use a perturbation method to solve these equations, so the key to the remainder of the analysis is the assumption of small-amplitude fluctuations. Suppose g_μ is sufficiently small that there exists a time scale T satisfying

$$T_C \ll T \ll T_F \equiv \frac{\langle F \rangle}{\left| g_\mu \frac{\partial \langle F \rangle}{\partial X_\mu} \right|} . \quad (2.4.3)$$

[Notice that these are the same conditions which were used before in equation (2.3.11) and Fig. 1.] Then according to (2.4.2) the variation δF generated by g_μ within a time T must remain much smaller than $\langle F \rangle$, and the right-hand side of (2.4.2) may be approximated by its first term. A higher approximation could be found by iteration. If we know the characteristics of the Stokes operator D/Dt --the unperturbed orbits--we may proceed to integrate (2.4.2) and substitute it into (2.4.1) to obtain an equation involving averaged quantities only:

$$\frac{D \langle F(X_\sigma, t) \rangle}{Dt} = \frac{1}{h} \frac{\partial}{\partial X_\mu} \left\langle h g_\mu(X_\sigma, t) \int_{t_0}^t dt' g_\nu[X'_\sigma(t'), t'] \frac{\partial \langle F[X'_\sigma(t'), t'] \rangle}{\partial X'_\nu} \right\rangle + o(g^4) . \quad (2.4.4)$$

Here $t - t_0 \ll T_F$ and $X'_\sigma(t')$ is that unperturbed orbit that passes through X_σ at $t' = t$.

In order to be able to evaluate the last derivative in (2.4.4) at X_σ rather than at $X'_\sigma(t')$, we should use natural coordinates suggested by the problem itself such that

$$\frac{DG_\mu}{Dt} = 0 , \quad (2.4.5)$$

as may be shown by a straightforward calculation of the difference between the two derivatives. This means that the problem is best solved in coordinates which are simply a parametrization of the unperturbed orbits, as was discovered in the preceding section. Then the integro-differential equation (2.4.4) is reduced to a pure differential equation:

$$\frac{D\langle F(X_\sigma, t) \rangle}{Dt} = \frac{1}{h} \frac{\partial}{\partial X_\mu} \left[h \left\langle g_\mu(X_\sigma, t) \int_{t_0}^t dt' g_\nu[X'_\sigma(t'), t'] \right\rangle \frac{\partial \langle F(X_\sigma, t) \rangle}{\partial X_\nu} \right] + O(g^4) . \quad (2.4.6)$$

The lowest-order nontrivial result appears now as a diffusion equation involving only second-order correlation functions of the field g_μ ; it may be written as

$$\frac{D\langle F \rangle}{Dt} \cong \nabla_\mu [D_{\mu\nu} \nabla_\nu \langle F \rangle] , \quad (2.4.7)$$

where

$$D_{\mu\nu}(X_\sigma, t) = h_\mu h_\nu \left\langle g_\mu(X_\sigma, t) \int_{-\infty}^t dt' g_\nu[X'_\sigma(t'), t'] \right\rangle . \quad (2.4.8)$$

It is the requirement in (2.4.3) that $t - t_0 \gg T_C$ which allows extension of the integration to $-\infty$ and makes $D_{\mu\nu}$ a function of t alone, eliminating any dependence upon initial conditions at t_0 . Since no correlation remains with conditions at t_0 , t may be taken as the beginning of another integration, and (2.4.7) may be thought of as being solved in this step-by-step manner to finally obtain $\langle F \rangle$ at any time, not limited from above by (2.4.3).

We can now show the relation of this derivation (QL) to that in the preceding section (FP). The derivative ∇_ν may be moved to the left in equation (2.4.7) to obtain

$$\frac{D\langle F \rangle}{Dt} \cong \nabla_\mu \nabla_\nu [D_{\mu\nu} \langle F \rangle] - \nabla_\mu [(\nabla_\nu D_{\mu\nu}) \langle F \rangle] . \quad (2.4.9)$$

By interchanging the dummy indices μ and ν , it is easily shown that the first term is determined entirely by the symmetric part of the diffusion tensor,

$$D_{\mu\nu}^S = \frac{1}{2} (D_{\mu\nu} + D_{\nu\mu}) , \quad (2.4.10)$$

but the second term will depend also on the antisymmetric part

$$D_{\mu\nu}^A = \frac{1}{2} (D_{\mu\nu} - D_{\nu\mu}) . \quad (2.4.11)$$

In the presence of magnetic fields, we must not expect that $D_{\mu\nu}^A$ should be zero, as it might otherwise be. With this information, (2.4.8) and (2.4.9) may be compared with (2.3.8), (2.3.9), and (2.3.12) to see that the results are expressed by formulae which are equivalent (except as noted in the following paragraph) no matter which derivation is used, even though a somewhat different meaning is attached to some of the symbols during the course of QL as opposed to FP. We have already seen that the same conditions upon the strength and coherence time of the fluctuations are required in either case, but there also seem to be two differences. First, FP makes the previously mentioned requirement on the interparticle separation in terms of fluctuation coherence wavelength, which apparently should still be of importance but is not made clear by QL; and, second, QL seems to justify omission of the condition (2.3.6).

The one actual difference in the formulae is that the FP coefficients include an additional averaging along unperturbed orbits of the QL coefficients. This will be unimportant as long as the statistical properties of the fluctuations remain fairly constant along each orbit,

$$\frac{D}{Dt} \langle g_\mu g_\nu \rangle \approx 0 ; \quad (2.4.12)$$

but this is equivalent to (2.2.2), and will be satisfied whenever either method can be used. Even insofar as the results are identical, the QL

derivation is seen to have the advantage that it clearly shows an equation of generalized diffusion form, (2.4.7), with coefficients (2.4.8) similar to (2.3.9) and so much easier to calculate than (2.3.8). In order to show this same form with the FP formulae one must be clever enough to rewrite (2.3.9), which is already explicitly symmetric, as the symmetric part of a nonsymmetric tensor whose divergence will be (2.3.8), thus making it apparent that the two terms can be combined. Equations (2.4.7) and (2.4.8) will be basic to the next chapter.

Chapter 3

RELATIVISTIC STOCHASTIC ACCELERATION: THE RESULTS

I. Terminology

The results of Chapter 2 will now be specifically applied to the motion of test particles of charge q and mass m in the electromagnetic fields of a weakly turbulent plasma. The treatment is to be relativistically valid, so position and momentum will be used as independent variables. The specification of fields implies the choice of a certain frame of observation; if some other frame is to be used, the fields must be Lorentz transformed. The physical content of the results must not depend on which reference frame is used, even though the description of the results may differ; what is seen by one observer as scattering may appear to another as acceleration. The relativistic covariance of the calculations will not be destroyed by the use of statistical methods as long as all averages are taken to be over ensembles.

Equations (2.2.5) will be the components in whatever coordinate system is chosen of the familiar vector equations

$$\frac{d\vec{x}}{dt} = \frac{\vec{p}}{m\gamma}, \quad (3.1.1)$$

$$\frac{d\vec{p}}{dt} = q \left[\vec{E}(\vec{x}, t) + \frac{1}{m\gamma c} \vec{p} \times \vec{B}(\vec{x}, t) \right], \quad (3.1.2)$$

where the total momentum alone determines the function

$$\gamma(p) = \sqrt{1 + (p/mc)^2}. \quad (3.1.3)$$

We write

$$\vec{E} = \vec{E}_0 + \Delta\vec{E}, \quad \vec{B} = \vec{B}_0 + \Delta\vec{B}$$

for the electric and magnetic fields, respectively, corresponding to the division between G_μ and g_μ in (2.2.5). The conditions (2.4.3) are fulfilled if typical magnitudes in the chosen frame of reference satisfy

$$q\Delta E T_C \ll p, \quad q\Delta B T_C \ll m\gamma c. \quad (3.1.4)$$

Note that for given field properties these conditions are more easily met by the relativistic particles which will be our principal concern.

The notation for the second-order correlation functions of the field fluctuations will be

$$R_{\alpha\beta}^{PQ}(\vec{\xi}, \tau) = \langle \Delta P_\alpha(\vec{x}, t) \Delta Q_\beta(\vec{x} + \vec{\xi}, t + \tau) \rangle, \quad (3.1.5)$$

where P and Q stand separately for either E or B , and α and β represent any directions in space. The assumption of a stationary, homogeneous process (as discussed in Chapter 2, Section II) is what makes this ensemble average independent of the coordinates \vec{x} and t . We define spectrum functions as Fourier transforms of these correlation functions:

$$R_{\alpha\beta}^{PQ}(\vec{\xi}, \tau) = \int d^3k \int d\omega e^{i(\vec{k} \cdot \vec{\xi} - \omega\tau)} S_{\alpha\beta}^{PQ}(\vec{k}, \omega), \quad (3.1.6)$$

$$S_{\alpha\beta}^{PQ}(\vec{k}, \omega) = (2\pi)^{-4} \int d^3\xi \int d\tau e^{-i(\vec{k} \cdot \vec{\xi} - \omega\tau)} R_{\alpha\beta}^{PQ}(\vec{\xi}, \tau). \quad (3.1.7)$$

These transform integrals, wherever they appear, will always be understood to have limits $-\infty$ and $+\infty$ without writing them explicitly.

Whenever a small-amplitude random-phase approximation is appropriate ("smallness in the second sense" of Chapter 2, Section I), the spectrum functions will be related to the Fourier transforms of the fields themselves by

$$S_{\alpha\beta}^{PQ}(\vec{k}, \omega) \delta^3(\vec{k} + \vec{k}') \delta(\omega + \omega') = \langle \tilde{P}_\alpha(\vec{k}, \omega) \tilde{Q}_\beta(\vec{k}', \omega') \rangle. \quad (3.1.8)$$

Faraday's Law supplies a relation between \vec{E} and \vec{B} which is independent of their origin; in terms of Fourier components it is

$$\omega \vec{B}(\vec{k}, \omega) = c \vec{k} \times \vec{E}(\vec{k}, \omega) . \quad (3.1.9)$$

From this would follow relations among the various $S_{\alpha\beta}^{PQ}$. These could be used to present all of the analysis in terms of electric fields alone (at the expense of extra vector algebra), but we prefer not to do so at this point for two reasons: We wish to show which field is responsible for each effect found, and we wish to allow for consideration of low-velocity modes where it would seem inappropriate to express \vec{B} in terms of a much smaller \vec{E} .

II. Field-Free Plasma

In order to discuss actual solutions, the fields \vec{E}_0 and \vec{B}_0 must be specified, and the logical point of departure is

$$\vec{E}_0 \equiv 0 , \quad \vec{B}_0 \equiv 0 . \quad (3.2.1)$$

We will briefly consider this simple case of an isotropic medium, even though no use will be made of it in our applications. One reason for this is to give the presentation some completeness; another is that it allows us to illustrate certain points without becoming entangled with others which will appear later.

The unperturbed orbits in this case are just straight lines,

$$\vec{x}'(t) = \vec{v}t , \quad \vec{p}'(t) = m\gamma\vec{v} = \text{constant} ; \quad (3.2.2)$$

we lose no generality by taking the initial position of the particle in question to be the origin of coordinates. The use of either spherical or Cartesian coordinates would be quite appropriate (even cylindrical or still other systems could be used); but let us choose Cartesian. Equation (3.1.1) involves only (constant) G_μ 's and the corresponding g_μ 's

are zero, and (3.1.2) has only g_μ 's with no G_μ 's. This causes all spatial derivatives to appear in the convective differential operator D/Dt , and leaves diffusion in momentum-space only,

$$\frac{\partial F}{\partial t} + \vec{v} \cdot \frac{\partial F}{\partial \vec{x}} = \frac{\partial}{\partial \vec{p}} \vec{D} \frac{\partial F}{\partial \vec{p}} . \quad (3.2.3)$$

(F is to be understood throughout the remainder of this work as a statistical expectation value, though the brackets $\langle \rangle$ used in the previous chapter will now be omitted.)

The diffusion tensor is found straightforwardly by using (2.4.8), (3.1.2), (3.1.5), and (3.2.2):

$$D_{\alpha\beta} = q^2 \int_{-\infty}^0 d\tau \left[R_{\alpha\beta}^{EE} + \frac{1}{c} \epsilon_{\beta\gamma\delta} v_\gamma R_{\alpha\delta}^{EB} + \frac{1}{c} \epsilon_{\alpha\gamma\delta} v_\gamma R_{\delta\beta}^{BE} + \frac{1}{c} \epsilon_{\alpha\gamma\delta} \epsilon_{\beta\mu\nu} v_\gamma v_\mu R_{\delta\nu}^{BB} \right] , \quad (3.2.4)$$

where all the correlation functions have the same arguments $(\vec{v}\tau, \tau)$ and the standard antisymmetric unit tensor $\epsilon_{\alpha\beta\gamma}$ has been used. This may also be written in terms of the spectrum functions by using (3.1.6); the first term, for example, becomes

$$\begin{aligned} q^2 \int_{-\infty}^0 d\tau R_{\alpha\beta}^{EE}(\vec{v}\tau, \tau) &= q^2 \int_{-\infty}^0 d\tau \int d^3k \int d\omega e^{i(\vec{k} \cdot \vec{v} - \omega)\tau} S_{\alpha\beta}^{EE}(\vec{k}, \omega) \\ &= q^2 \int d^3k \int d\omega S_{\alpha\beta}^{EE}(\vec{k}, \omega) \left[\pi \delta(\vec{k} \cdot \vec{v} - \omega) - P \frac{1}{\vec{k} \cdot \vec{v} - \omega} \right] , \end{aligned} \quad (3.2.5)$$

where $\delta(\)$ is the Dirac delta-function and P denotes Cauchy principal-value integration. By changing the signs of the dummy variables \vec{k} and ω and using the symmetry property (B.2) we may see that the delta-function contributes only to the symmetric part of the diffusion tensor and the principal-value only to the antisymmetric part:

$$D_{\alpha\beta}^S = \pi q^2 \int d^3k \left[S_{\alpha\beta}^{EE} + \frac{1}{c} \epsilon_{\beta\gamma\delta} v_\gamma S_{\alpha\delta}^{EB} + \frac{1}{c} \epsilon_{\alpha\gamma\delta} v_\gamma S_{\delta\beta}^{BE} + \frac{1}{2} \epsilon_{\alpha\gamma\delta} \epsilon_{\beta\mu\nu} v_\gamma v_\mu S_{\delta\nu}^{BB} \right] \quad (3.2.6)$$

with arguments $S(\vec{k}, \vec{k} \cdot \vec{v})$, and

$$D_{\alpha\beta}^A = -iq^2 p \int d\omega \int d^3k (\vec{k} \cdot \vec{v} - \omega)^{-1} \left[S_{\alpha\beta}^{EE} + \frac{1}{c} \epsilon_{\beta\gamma\delta} v_\gamma S_{\alpha\delta}^{EB} + \frac{1}{c} \epsilon_{\alpha\gamma\delta} v_\gamma S_{\delta\beta}^{BE} + \frac{1}{2} \epsilon_{\alpha\gamma\delta} \epsilon_{\beta\mu\nu} v_\gamma v_\mu S_{\delta\nu}^{BB} \right] \quad (3.2.7)$$

with arguments $S(\vec{k}, \omega)$. Thus the diffusion of any particle is caused by waves traveling at or near the velocity of that particle; that is, the waves must appear at or near zero frequency to an observer moving with the particle. Equation (B.4) confirms that $D_{\mu\nu}^S$ and $D_{\mu\nu}^A$ are both real.

As we remarked in Chapter 2, a further step one would eventually take is to classify the wave modes that could be present (which would be fairly simple in this case, but much worse in the following section) and to relate the energy of each to its particular field amplitudes, so that the diffusion tensor could be written explicitly in terms of these wave mode energy densities. For the most general conditions at the distant boundaries the results would still be quite complicated, but if these boundary conditions are isotropic or (more to the point) if they are irrelevant because the turbulence is a locally generated quasi-steady state, then the wave spectra will have the same isotropy as the plasma itself. The same must be true of the diffusion tensor, and its most general dependence upon the momentum would appear in this case to be of the form

$$D_{\alpha\beta} = D_1 (p^2) \delta_{\alpha\beta} + D_2 (p^2) p_\alpha p_\beta + D_3 (p^2) \epsilon_{\alpha\beta\gamma} p_\gamma, \quad (3.2.8)$$

with D_1 and D_2 determined by (3.2.6) and D_3 by (3.2.7). Here the standard symmetric unit tensor (Kronecker symbol) $\delta_{\alpha\beta}$ has also been introduced.

III. Uniformly Magnetized Plasma

The next simplest case is that where \vec{E}_0 and \vec{B}_0 are constant and uniform, but $|\vec{E}_0| < |\vec{B}_0|$ and $\vec{E}_0 \cdot \vec{B}_0 = 0$; the latter condition is often a good approximation, for a highly conductive plasma cannot ordinarily support any steady, large-scale electric field parallel to \vec{B}_0 . Once the fields are assumed uniform and perpendicular, the smaller of them can always be removed by a Lorentz transformation; since the electric field is here being taken smaller, we shall suppose that we have already chosen that frame of observation where $\vec{E}_0 = 0$. Along with (3.1.4), it is now also required that $\Delta B \ll B_0$ for the following analysis to be valid.

It is natural here to use either cylindrical or spherical coordinates in momentum space, with polar axis in the direction of $\vec{B}_0 = B_0 \hat{z}$. Each particle may be characterized by a natural gyrofrequency and gyroradius,

$$\Omega = \Omega_0/\gamma = qB_0/m\gamma c, \quad r_g = v_\perp/\Omega,$$

and it will prove convenient to introduce complex transverse coordinates in position space, such that any vector \vec{A} is defined to have components

$$A_\pm = A_x \pm iA_y.$$

(Some authors prefer a different definition; for further properties see Appendix A.) Then the unperturbed orbit of any particle may be written as

$$x'_\pm(t) = \pm i r_g e^{\mp i(\Omega t - \phi_0)}, \quad (3.3.1)$$

$$z'(t) = v_\parallel t, \quad (3.3.2)$$

$$p'_\perp(t) = p_\perp, \quad p'_\parallel(t) = p_\parallel \quad (3.3.3a)$$

or

$$p'(t) = p, \quad \theta'(t) = \theta, \quad (3.3.3b)$$

and

$$\phi'(t) = \phi_0 - \Omega t ; \quad (3.3.4)$$

all quantities other than t on the right-hand sides of these equations are constants of the motion. In this standard notation, ϕ is the azimuthal angle in momentum space and θ , the polar angle, is also the pitch angle of the helical path followed by the particle. Again, the assumed uniformity of field properties allows us to choose the origin of spatial coordinates at the initial position of the guiding center of the particle in question.

The coordinate ϕ is relatively uninteresting; it is true that it must be used in (3.3.1) in order to properly evaluate the correlation functions, but for the purpose of following the momentum-space diffusion, the other two coordinates are more important if times $t \gg \Omega^{-1}$ are being considered. Thus in a Chew-Goldberger-Low type approximation (discussed at greater length by Kennel and Engelmann⁷⁹), we limit our attention to \bar{F} , the phase-independent part of F . The lowest-order equation obtained by averaging (2.4.7) has the same form, but the subscripts now range only from 1 to 5, F is replaced by \bar{F} , and $D_{\mu\nu}$ becomes

$$\bar{D}_{\mu\nu} = \frac{1}{2\pi} \int_0^{2\pi} d\phi D_{\mu\nu} . \quad (3.3.5)$$

Accompanying the elimination of the cyclic coordinate ϕ is a disinterest in the fluctuating part of x_{\pm} , in favor of following only the guiding center motion. This change must be made, in fact, to satisfy (2.4.5). Actually, we define a pseudo-guiding center (which never differs at any time from the true guiding center by more than a small fraction of a gyroradius) of the projected motion in the x - y plane by using only the unperturbed field;

$$x_{g+} = x_+ + v_+ / i\Omega . \quad (3.3.6)$$

The equation of motion for this variable is

$$\frac{dx_{g+}}{dt} = \frac{1}{m\gamma} p_{+} + \frac{1}{i\Omega} \frac{dp_{+}}{dt} \quad (3.3.7a)$$

$$= \frac{c}{iB_0} \Delta E_{+} + \frac{v_{\parallel}}{B_0} \Delta B_{+} - \frac{v_{+}}{B_0} \Delta B_z, \quad (3.3.7b)$$

and for the others we have

$$\frac{dz}{dt} = v_{\parallel} \quad (3.3.8)$$

and

$$\frac{dp_{\perp}}{dt} = q\Delta E_t + \frac{i\Omega}{B_0} p_{\parallel} \Delta B_{\theta}, \quad (3.3.9)$$

$$\frac{dp_{\parallel}}{dt} = q\Delta E_z - \frac{i\Omega}{B_0} p_{\perp} \Delta B_{\theta}, \quad (3.3.10)$$

or

$$\frac{dp}{dt} = q(\sin \theta \Delta E_t + \cos \theta \Delta E_z), \quad (3.3.11)$$

$$\frac{d\theta}{dt} = q \left(\frac{\cos \theta}{p} \Delta E_t - \frac{\sin \theta}{p} \Delta E_z + \frac{i\Omega}{B_0} \Delta B_{\theta} \right). \quad (3.3.12)$$

Here abbreviations have been introduced for the quantities

$$\Delta E_t = \frac{1}{2} \left[e^{-i\phi} \Delta E_{+} + e^{i\phi} \Delta E_{-} \right], \quad \Delta B_{\theta} = \frac{1}{2} \left[e^{-i\phi} \Delta B_{+} - e^{i\phi} \Delta B_{-} \right]. \quad (3.3.13)$$

It may be remarked here that both parts of (3.3.7) serve a purpose; (3.3.7a) assures that $(\partial/\partial x_{g+})(dx_{g+}/dt) = 0$, but since it involves the coordinate ϕ it cannot be used as a G_μ [as can (3.3.8)] to put all spatial variation into the convective derivative $D\bar{F}/Dt$. Rather, (3.3.7b) must be used as a g_μ to calculate a position space diffusion. Then equation (2.4.7) takes the form

$$\frac{\partial \bar{F}}{\partial t} + v_z \frac{\partial \bar{F}}{\partial z} = \nabla_\mu \left[\bar{D}_{\mu\nu} \nabla_\nu \bar{F} \right], \quad (3.3.14)$$

with μ and ν now enumerating only the four variables x_{g+} and x_{g-} (or x_g and y_g) and p_\perp and p_\parallel (or p and θ).

It may be seen from (3.3.7b) and (3.3.9) through (3.3.13) that the calculation of the diffusion coefficients of (3.3.14) involves in every case one of a certain family of integrals which are evaluated in Appendix C, so that we have only to put in proper values for subscripts, parameters, and multiplicative constants to obtain complete formulae for $\bar{D}_{\mu\nu}$. Consider first the momentum-space diffusion alone (the spatial diffusion will be taken up in Section VII). This depends on the three field combinations ΔE_z , ΔE_t and ΔB_θ , and so on the nine quantities

$$\begin{aligned} \bar{\Gamma}_{zz} &= q^2 \frac{1}{2\pi} \int_0^{2\pi} d\phi \left\langle \Delta E_z[\vec{x}, t] \int_{-\infty}^t dt' \Delta E_z[\vec{x}'(t'), t'] \right\rangle \\ &= q^2 \int d^3k \int d\omega \sum_n J_n^2(k_\perp r_g) S_{zz}^{EE}(\vec{k}, \omega) \pi \delta(k_\parallel v_\parallel + n\Omega - \omega), \end{aligned} \quad (3.3.15)$$

$$\begin{aligned} \bar{\Gamma}_{tt} &= \frac{1}{4} q^2 \int d^3k \int d\omega \sum_n \left\{ 2J_{n+1}^2 S_{+-}^{EE} + J_{n-1} J_{n+1} \left[e^{-2i\varphi} S_{++}^{EE} + e^{2i\varphi} S_{--}^{EE} \right] \right\} \\ &\quad \pi \delta(k_\parallel v_\parallel + n\Omega - \omega), \end{aligned} \quad (3.3.16)$$

$$\bar{\Gamma}_{\theta\theta} = \frac{\Omega^2 p^2}{4B_o^2} \int d^3k \int d\omega \sum_n \left\{ 2J_{n+1}^2 S_{+-}^{BB} - J_{n-1} J_{n+1} \left[e^{-2i\varphi} S_{++}^{BB} + e^{2i\varphi} S_{--}^{BB} \right] \right\}$$

$$\pi \delta(k_{\parallel} v_{\parallel} + n\Omega - \omega) , \quad (3.3.17)$$

$$\bar{\Gamma}_{zt} = \bar{\Gamma}_{tz} = 0 , \quad (3.3.18)$$

$$\bar{\Gamma}_{z\theta} = \bar{\Gamma}_{\theta z} = 0 , \quad (3.3.19)$$

$$\bar{\Gamma}_{\theta t} = \frac{iq\Omega p}{4B_o} \int d^3k \int d\omega \sum_n \left\{ J_{n-1}^2 S_{+-}^{BE} - J_{n+1}^2 S_{-+}^{BE} + J_{n-1} J_{n+1} \left[e^{-2i\varphi} S_{++}^{BE} - e^{2i\varphi} S_{--}^{BE} \right] \right\}$$

$$\left[\pi \delta(k_{\parallel} v_{\parallel} + n\Omega - \omega) \pm p \frac{1}{k_{\parallel} v_{\parallel} + n\Omega - \omega} \right] \quad (3.3.20)$$

The same arguments apply to the Bessel and spectrum functions throughout, and φ is the azimuthal angle in k-space.

Straightforward use of either the cylindrical or the spherical coordinate system gives two sets of four $\bar{D}_{\mu\nu}$'s which are linear combinations of the $\bar{\Gamma}_{\mu\nu}$'s. These are exhibited in Appendix D; unfortunately, they do not make the physical meaning of the process very clear. But by allowing simultaneous use of both coordinate systems, we find that when everything is written out the relations

$$\frac{\partial}{\partial \theta} = p_{\parallel} \frac{\partial}{\partial p_{\perp}} - p_{\perp} \frac{\partial}{\partial p_{\parallel}} \quad (3.3.21a)$$

and

$$p \frac{\partial}{\partial p} = p_{\perp} \frac{\partial}{\partial p_{\perp}} + p_{\parallel} \frac{\partial}{\partial p_{\parallel}} \quad (3.3.21b)$$

may be used to manipulate the result into the very suggestive form

$$\begin{aligned}
\left(\frac{D\bar{F}}{Dt}\right)_{pp} = & \frac{\partial}{\partial p_{\parallel}} \left[\bar{\Gamma}_{zz} \frac{\partial \bar{F}}{\partial p_{\parallel}} \right] + \frac{1}{p_{\perp}} \frac{\partial}{\partial p_{\perp}} \left[p_{\perp} \bar{\Gamma}_{tt} \frac{\partial \bar{F}}{\partial p_{\perp}} \right] + \frac{1}{p^2 \sin \theta} \frac{\partial}{\partial \theta} \left[\sin \theta \bar{\Gamma}_{\theta\theta} \frac{\partial \bar{F}}{\partial \theta} \right] \\
& + \frac{\partial}{\partial p_{\parallel}} \left[\bar{\Gamma}_{zt} \frac{\partial \bar{F}}{\partial p_{\perp}} \right] + \frac{1}{p_{\perp}} \frac{\partial}{\partial p_{\perp}} \left[p_{\perp} \bar{\Gamma}_{tz} \frac{\partial \bar{F}}{\partial p_{\parallel}} \right] + \frac{\partial}{\partial p_{\parallel}} \left[\bar{\Gamma}_{z\theta} \frac{1}{p} \frac{\partial \bar{F}}{\partial \theta} \right] \\
& + \frac{1}{p \sin \theta} \frac{\partial}{\partial \theta} \left[\sin \theta \bar{\Gamma}_{\theta z} \frac{\partial \bar{F}}{\partial p_{\parallel}} \right] + \frac{1}{p_{\perp}} \frac{\partial}{\partial p_{\perp}} \left[p_{\perp} \bar{\Gamma}_{t\theta} \frac{1}{p} \frac{\partial \bar{F}}{\partial \theta} \right] \\
& + \frac{1}{p \sin \theta} \frac{\partial}{\partial \theta} \left[\sin \theta \bar{\Gamma}_{\theta t} \frac{\partial \bar{F}}{\partial p_{\perp}} \right]. \tag{3.3.22}
\end{aligned}$$

Here each partial derivative retains the meaning it would normally have in the standard coordinate systems. This is the same in some respects as if θ were formally an independent coordinate along with p_{\perp} and p_{\parallel} , with $\bar{\Gamma}_{\mu\nu}$ playing the role of a diffusion tensor in this artificially three-dimensional momentum space, and makes it clearer than would (3.3.9)-(3.3.12) that the three fields ΔE_z , ΔE_t , and ΔB_{θ} are uniquely bound to producing changes in p_{\parallel} , p_{\perp} , and θ , respectively.

Again, as in Section II, we may remark that this solution will in general depend on boundary conditions. But if the fluctuation spectrum is locally generated, its statistical properties will have the same gyro-tropic symmetry as the background plasma; the consequences of this are given in Appendix B. All integrations over the azimuthal angle φ in k -space may then be carried out on the basis of this symmetry alone; in the particular case of (3.3.15)-(3.3.20), this does not in itself cause any terms to vanish.

IV. Parallel and Cyclotron Acceleration

The next three sections are devoted to the elucidation of the results in Section III by the study of several special limits. As the first case, let us consider the effect of fluctuating electric fields alone as

was done in SA. Then $\bar{\Gamma}_{zz}$ and $\bar{\Gamma}_{tt}$ are the nonvanishing quantities whose meaning is to be interpreted. First of all, a constant or asymptotic solution \bar{F} independent of p_{\perp} and p_{\parallel} cannot be allowed in the global sense because it is not normalizable. The next best way to understand the characteristics of these terms is to inquire into the existence of a self-similar solution. If $\bar{\Gamma}_{zz}$ and $\bar{\Gamma}_{tt}$ are constant, the problem reduces to a simple diffusion equation and there is indeed such a solution,

$$\bar{F} \sim t^{-3/2} \exp \left[- \frac{p_{\parallel}^2}{4\bar{\Gamma}_{zz} t} - \frac{p_{\perp}^2}{4\bar{\Gamma}_{tt} t} \right]. \quad (3.4.1)$$

This has the appearance of a Maxwellian distribution for which the temperature is increasing linearly with time, but of course it is actually cut off much more strongly than a true Maxwellian at high energies, because for relativistic particle of energy $W = \gamma mc^2$ this varies as e^{-W^2} rather than e^{-W} . We may already conclude from this that (a) it is perfectly normal that the particles receive a net flow of energy from the field fluctuations, but (b) this can be of interest for generating relativistic particles only if there is some way of making the effective value of $\bar{\Gamma}$ increase with energy.

Suppose next that a spectrum is considered for which the wave numbers of the most important waves satisfy $k_{\perp} r_g \ll 1$ for the particles of interest. Then the predominant part of the coefficients will be

$$\bar{\Gamma}_{zz} \rightarrow \pi q^2 \int d^3 k S_{zz}^{EE}(\vec{k}, k_{\parallel} v_{\parallel}) \quad (3.4.2)$$

and

$$\bar{\Gamma}_{tt} \rightarrow \frac{1}{2} \pi q^2 \int d^3 k S_{+-}^{EE}(\vec{k}, k_{\parallel} v_{\parallel} - \Omega). \quad (3.4.3)$$

In this limit there are clearly two independent effects: acceleration by the electric fields parallel to \vec{B}_0 at zero frequency (as seen by an observer moving with the particle), and acceleration by the transverse fields at cyclotron resonance. The second of these processes has a distinct advantage over the first for the acceleration of relativistic particles, for $\bar{\Gamma}_{zz}$ depends mainly upon pitch angle whereas the variation of Ω with energy can cause $\bar{\Gamma}_{tt}$ to increase with energy if, as is quite reasonable by inference from turbulence in ordinary fluids, we should have a spectrum function which decreases toward higher frequencies. On the other hand, the simultaneous small-gyroradius and non-relativistic limit makes $\bar{\Gamma}_{tt}$ independent of p_{\perp} ; then we recover (2.26) and (3.20) of SA, the consequences of which have already been discussed there.

For a given wave spectrum with its maximum important k_{\perp} , there is (assuming relativistic particles again now) a characteristic particle energy

$$W(k_{\perp}) = \frac{qB_0}{k_{\perp}} \quad (3.4.4)$$

above which $k_{\perp} r_g \gtrsim 1$. We suggest that this will superimpose more or less of a cutoff on the $\bar{\Gamma}$ that the wave spectrum would otherwise give above this energy. To see this consider the identity

$$\sum_n J_n^2(x) = 1, \quad (3.4.5)$$

which states that a constant total weight of unity is being assigned by the summation in (3.3.15) or the first term of (3.3.16) regardless of the size of $k_{\perp} r_g$, but redistributed to different parts of the spectrum. In the most usual circumstance of a spectrum which falls off in frequency, this transfer of weight from lower to higher harmonics will decrease $\bar{\Gamma}$ even though the total frequency range being spanned by the larger number of harmonics may remain about the same in the relativistic case.

We may also use

$$\sum_n J_{n-1}(x) J_{n+1}(x) = 0 \quad (3.4.6)$$

in the same way to suggest that the terms in which this combination of Bessel functions appears will never attain much relative importance for any value of $k_{\perp} r_g$, except as a result of some unusual or peculiar form for the spectra. This increases the probability that insights gained from consideration of the small-gyroradius limit will remain more or less generally true.

This is an appropriate place to comment on the connection of our work to that of Puri.⁶⁵ He has used a different mathematical method to investigate fluctuations in the time domain alone; that is, in the strong limit where both $k_{\perp} r_g \ll 1$ and $k_{\parallel} v T_C \ll 1$. This simplifies the analysis in that the departure of a particle from its original orbit does not alter the field strength acting upon it. In our notation one could write

$$S(\vec{k}, \omega) \cong \delta^3(\vec{k}) \Phi(\omega) , \quad (3.4.7)$$

and immediately perform all k -integrations to obtain

$$\bar{\Gamma}_{zz} \rightarrow \pi q^2 \Phi(0) , \quad (3.4.8)$$

$$\bar{\Gamma}_{tt} \rightarrow \frac{1}{2} \pi q^2 \Phi(\Omega) , \quad (3.4.9)$$

etc. But Puri has been able to show, under appropriate assumptions about statistical properties, that such formulae can remain valid for arbitrary field amplitudes.

V. Magnetic Scattering

The second special case of momentum space diffusion is complementary to the first. As was pointed out elsewhere,⁷² Faraday's Law may be used to write the ratio of magnetic and electric forces from a Fourier-analyzed disturbance as

$$\frac{|\vec{v} \times \vec{B}|}{c|\vec{E}|} = \frac{|\vec{v} \times (\vec{k} \times \vec{E})|}{\omega|\vec{E}|} \quad (3.5.1)$$

This indicates that the electric aspect of the wave is more important for all purely longitudinal waves and for all transverse waves with phase velocity $(\omega/k) \gg v$, so that the approximation of the previous section would be appropriate. But the magnetic field will be more important for most transverse waves with $(\omega/k) \ll v$, and the treatment of the present section may be applied to find their effect. Hydromagnetic waves acting upon relativistic particles would be an example of this case. (The criterion as roughly described here would depend on the frame of observation being used; discussion of this point is given in an unpublished report.⁷⁶)

A purely magnetic field cannot change the energy of a particle so the effect of magnetic fluctuations alone, when their associated electric induction fields may be neglected, is limited to the scattering in pitch angle described by (3.3.17). This too is a cyclotron-resonance effect; its simple form for $k_{\perp} r_g \ll 1$ is

$$\bar{\Gamma}_{\theta\theta} \rightarrow \frac{\pi \Omega^2 p^2}{2B_0^2} \int d^3k S_{+-}^{BB}(\vec{k}, k_{\parallel} v_{\parallel} - \Omega) \quad (3.5.2)$$

Just as with the cyclotron acceleration, a higher-energy relativistic particle can resonate with a lower-frequency (and so likely stronger) wave. This may partially or wholly offset the explicit factor $\Omega^2 \sim \gamma^{-2}$ and enhance the likelihood that this magnetic scattering will become relatively more important, as particle energy is increased, than Coulomb scattering from individual background particles.

To be more specific, magnetic fluctuations of typical relative amplitude $\delta = \Delta B/B_0$, coherence time T_C , and spectrum falling off as ω^{-n} above T_C^{-1} will cause scattering through an angle of order unity in a time

$$T_B \sim p^2 \bar{\Gamma}_{\theta\theta}^{-1} \sim \frac{1 + (\Omega_C)^n}{\delta^2 \Omega_C^2 T_C} . \quad (3.5.3)$$

For comparison, the time for substantial deflection by Coulomb scattering,⁸⁰ with account taken of the relativistic correction to the Rutherford cross section,⁸¹ is about

$$T_{\text{Cou}} \sim \frac{m^2 c^3 \beta^3 \gamma^2}{16\pi q_o^2 n_o \ln \Lambda} , \quad (3.5.4)$$

where $\beta = v/c$ and the density n_o , charge q_o and $\ln \Lambda$ pertain to the background plasma. Magnetic scattering will be the more important process if

$$\frac{T_B}{T_{\text{Cou}}} \sim \frac{16\pi q_o^2 n_o \ln \Lambda [1 + (\Omega_C)^n]}{c \delta^2 B_o^2 \beta^3 T_C} \lesssim 1 . \quad (3.5.5)$$

If we suppose the background particles to be singly charged, and $\ln \Lambda \sim 20$, this becomes

$$10^{46} \frac{mB_o}{n_o} \delta^2 \gtrsim \frac{1 + (\Omega_C T_C / \gamma)^n}{\beta^3 \Omega_C T_C} , \quad (3.5.6)$$

or for sufficiently relativistic particles

$$(\Delta B)^2 T_C \gtrsim 10^{-26} n_o . \quad (3.5.7)$$

This last condition would be satisfied virtually anywhere, and represents such an extreme as to be uninteresting. For a practical test one must use (3.5.6); this equation may be interpreted graphically as in Fig. 2, which is drawn for the case $n = 2$. In a particular physical problem involving a particular species one generally knows m , n_0 , and B_0 (and thus $\Omega_0 = qB_0/mc$), so the scales are determined for all three variables, δ , T_C , and (kinetic) energy $E = W - mc^2 = (\gamma - 1)mc^2$. For given values of δ and T_C , the graph then determines a minimum energy above which magnetic scattering may be important, but below which it gives way to Coulomb scattering. Conversely, given some energy E and time T_C , the graph determines a minimum amplitude δ above which magnetic scattering dominates for particles of this energy; or, given values of δ and E , one may find what values of T_C , if any, will accomplish this.

Consider the example of scattering of energetic protons in the interstellar medium. This is important to the isotropy of cosmic rays as well as to some attempts to explain their acceleration (Fermi traps^{38,39}). Using $n_0 \sim 1 \text{ cm}^{-3}$ and $B_0 \sim 5 \times 10^{-6}$ gauss, the approximate locations of $\delta = 1$ (line P) and $T_C = 1 \text{ sec}$ (line Q) are found. We also note at the top the time scales given by the background self-collision frequency, $\nu^{-1} \sim 10^4 \text{ sec}$, and by the transit time of hydromagnetic waves over the scale of the known inhomogeneity represented by the H II "clouds," $L/V_A \sim 1 \text{ lightyear}/10^{-4.5} \text{ c} \sim 10^{12} \text{ sec}$. Then take for illustration the point R, for which $\delta = 10^{-1}$ and $T_C = 10^{10} \text{ sec} \approx 300 \text{ yr}$. If such fluctuations are present, they will scatter protons more strongly than will Coulomb collisions for all energies above 1 MeV, and a scattering time $T_B \sim 10^{11.7} \text{ sec} \sim 10^4 \text{ yr}$ will apply up to a maximum energy of about 10^{18} eV , i.e., essentially for all cosmic rays. Similarly, reasonable parameters can lead to the expectation of magnetic scattering on an interesting scale for electrons participating in Type IV solar radio bursts, for "solar cosmic rays" in the interplanetary medium, or for energetic particles in distant objects such as supernova remnants, radio galaxies, and quasars. We shall consider in succeeding chapters what effect this may have, in combination with other processes, in determining natural energetic-particle spectra.

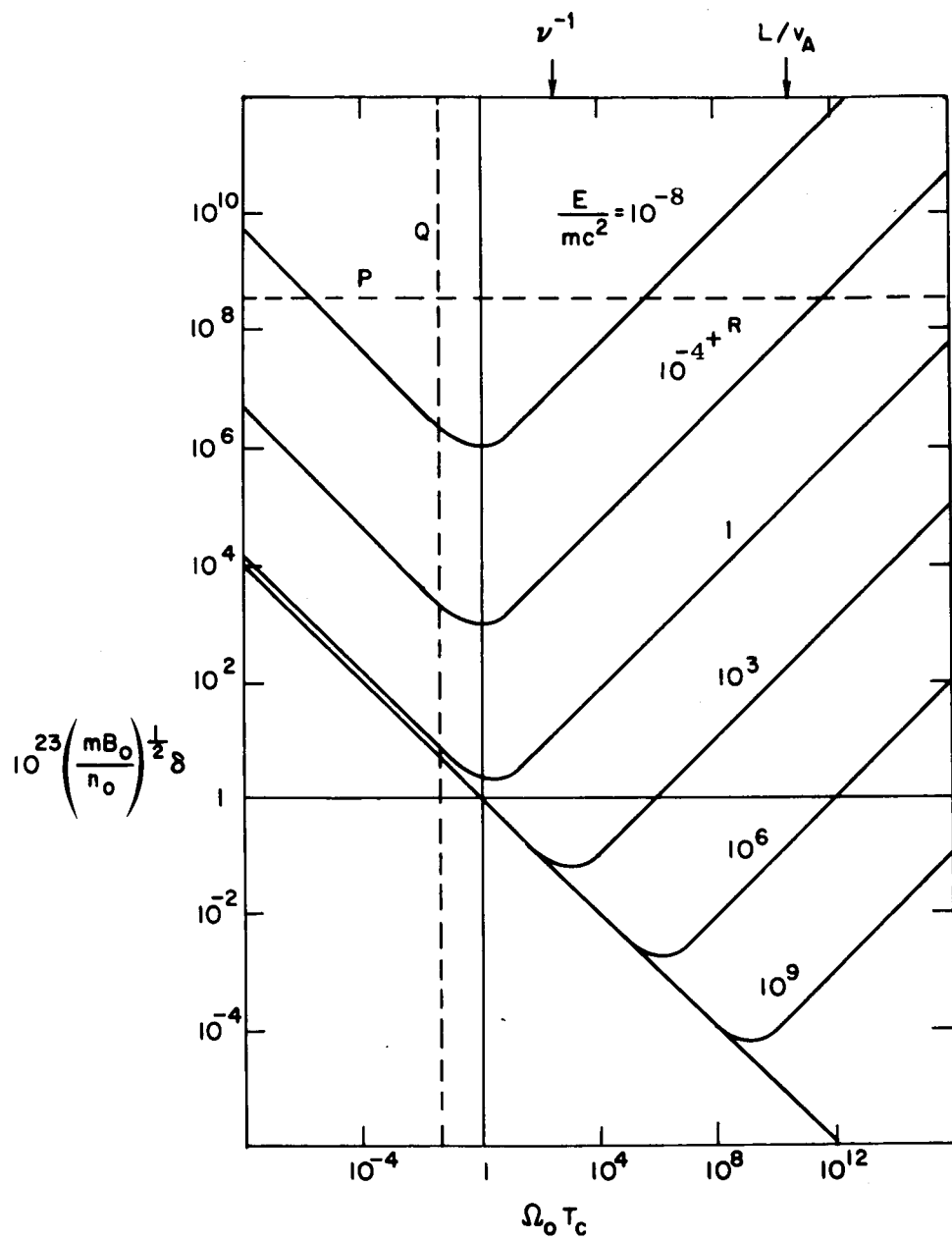


FIG. 2. THE FUNCTION OF EQUATION (3.5.6).

One case among those mentioned that has already attracted extensive study is that of the interplanetary medium. The importance of magnetic scattering in the diffusive propagation of energetic particles through the solar wind has been recognized for some time,⁸² and this case is also exceptional in that space probe techniques offer the possibility of direct observation of both particles and fields. Coleman has recently reported a study of such measurements on the magnetic field by Mariner 2. These show a power spectrum of magnetic fluctuations varying with frequency over the measured range $1.5 \text{ min} < (2\pi/\omega) < 8 \text{ hr}$ in a way that may be represented roughly by $S^{BB} \sim \omega^{-n}$, with $n \simeq 1$ for the low and medium frequencies in this range but increasing to perhaps $n \sim 2$ for the higher frequencies.⁸³ On the theoretical side, the prospect of using these measurements to build a more detailed picture of the energetic-particle propagation has led to the replacement of Parker's simple model of a single scattering⁸⁴ by the more general treatment of Jokipii.⁸⁵ When differences in notation are removed, the latter's results are equivalent to those that would be obtained from our more general theory in the limit of a zero-frequency spectrum,

$$S(\vec{k}, \omega) = S'(\vec{k}) \delta(\omega) . \quad (3.5.8)$$

We conclude this section by pointing out that, insofar as the idealization of purely magnetic fluctuations is realized, the scattering they cause must be isotropic. This is a consequence only of reducing (3.3.22) to

$$\frac{\partial \bar{F}}{\partial t} = \frac{1}{p^2 \sin \theta} \frac{\partial}{\partial \theta} \left[\sin \theta \bar{\Gamma}_{\theta\theta} \frac{\partial \bar{F}}{\partial \theta} \right] , \quad (3.5.9)$$

with no other requirement being made of $\bar{\Gamma}_{\theta\theta}$ except that

$$\sin \theta \bar{\Gamma}_{\theta\theta} \bar{F} \frac{\partial \bar{F}}{\partial \theta} \rightarrow 0$$

at the limits $\theta = 0, \pi$. The proof consists of defining the quantities

$$\bar{F}_{av}(p, t) = \frac{1}{2} \int_0^\pi d\theta \sin\theta \bar{F}(p, \theta, t), \quad (3.5.10)$$

$$I(p, t) = \frac{1}{2} \int_0^\pi d\theta \sin\theta (\bar{F} - \bar{F}_{av})^2, \quad (3.5.11)$$

and then computing

$$\begin{aligned} \frac{\partial I}{\partial t} &= \int_0^\pi d\theta (\bar{F} - \bar{F}_{av}) \frac{1}{p} \frac{\partial}{\partial \theta} \left[\sin\theta \bar{\Gamma}_{\theta\theta} \frac{\partial \bar{F}}{\partial \theta} \right] \\ &= - \frac{1}{2} \int_0^\pi d\theta \sin\theta \bar{\Gamma}_{\theta\theta} \left(\frac{\partial \bar{F}}{\partial \theta} \right)^2. \end{aligned} \quad (3.5.12)$$

Since $\bar{\Gamma}_{\theta\theta}$ is positive [remember that it is just twice the Fokker-Planck coefficient $\langle (\Delta\theta)^2 / \Delta t \rangle$], $I(p, t)$ must be monotonically decreasing (possibly at a different rate for different p) toward the lower bound $I \equiv 0$ corresponding to $\partial \bar{F} / \partial \theta = 0$. Thus the only well-behaved asymptotic solution under the influence of this process alone is that for which the distribution function becomes isotropic in momentum space. This statement is slightly stronger and more accurate than the brief converse argument used by Jokipii,⁸⁵ who noted that $\partial \bar{F} / \partial \theta = 0$ describes a stationary solution and concluded that this requires the two Fokker-Planck terms to combine into the form (3.5.9).

VI. Magnetic Pumping

We give now an example in which the magnetic and electric aspects of a particular type of wave are both taken into account together; some of the ideas in this section have been published in brief-communication form.⁸⁶ The concept of magnetic pumping is simple and familiar to many plasma physicists; as a practical means of heating particles, it is

generally presented in terms of a coherent applied magnetic field whose energy may be absorbed because of a finite collision frequency.⁸⁷ Here we use the term "stochastic magnetic pumping" to discuss situations where the frequency of collisions between particles is very small but where energy may still be transferred from pumping-type fields to particles because these are taken to be the random fields associated with the turbulence spectrum of a nonthermal plasma. Small but finite gyroradii must be used to find this effect, and the method differs from that of the preceding sections in that the model pumping waves used cannot be analyzed into statistically independent plane waves. Thus the interaction of particles with low-frequency transverse fields which will be found here is not in contradiction to the comments following (3.4.3), which apply to plane waves.

Consider a spectrum of magnetic compressional waves for which the convergence of field lines is more important than their curvature, and which are idealized in cylindrical geometry so that they resemble moving magnetic mirrors. Then a Taylor expansion of the change in longitudinal magnetic field (the "amount of squeezing") about the symmetry axis contains only even powers of the radius:

$$\Delta B_z(x_+, x_-, z, t) = B_{z0}(z, t) + B_{z2}(z, t)x_+x_- + \dots \quad (3.6.1)$$

The use of low-order terms in this series will give a "paraxial" approximation. The corresponding expansion for the transverse magnetic field will have odd powers of the radius, and by requiring the field to have zero divergence we can easily relate it to the longitudinal field:

$$\Delta B_+(x_+, x_-, z, t) = -\frac{1}{2} \frac{\partial B_{z0}}{\partial z} x_+ - \frac{1}{4} \frac{\partial B_{z2}}{\partial z} x_+^2 x_- - \dots \quad (3.6.2)$$

Likewise, the transverse electric field can be found from Faraday's Law to be

$$\Delta E_+(x_+, x_-, z, t) = -\frac{i}{2c} \frac{\partial B_{z0}}{\partial t} x_+ - \frac{i}{4c} \frac{\partial B_{z2}}{\partial t} x_+^2 x_- - \dots \quad (3.6.3)$$

But the longitudinal electric field is undetermined--it depends upon what currents may be flowing along the magnetic field lines--so we shall leave it as an independent quantity for the moment and later discuss what its nature is likely to be. But we do suppose it to be expanded in powers of the radius as was done in (3.6.1).

For particles moving near the axis of the fields just described, the lowest-order part (in gyroradius) of (3.3.9) and (3.3.10) will be

$$\frac{dp_{\perp}}{dt} = \frac{p_{\perp}}{2B_0} \left(\frac{\partial}{\partial t} + v_{\parallel} \frac{\partial}{\partial z} \right) B_{z0}(z, t) , \quad (3.6.4)$$

$$\frac{dp_{\parallel}}{dt} = qE_{z0}(z, t) - \frac{p_{\perp}^2}{2m\gamma B_0} \frac{\partial B_{z0}}{\partial z}(z, t) . \quad (3.6.5)$$

When the Fourier transforms are used, the derivatives acting on B_{z0} will be replaced by multiplicative factors, and the spectrum functions will be evaluated at $\omega = k_{\parallel} v_{\parallel}$. But (3.6.4) will have the factor $\omega - k_{\parallel} v_{\parallel}$, which will cause its contributions to vanish--that is, we have in the first order the expected result that

$$\mu = \frac{p_{\perp}^2}{2mB_z} \quad (3.6.6)$$

is an adiabatic invariant. Thus (3.3.22) reduces in this case to

$$\left(\frac{D\bar{F}}{Dt} \right)_{pp} = \frac{\partial}{\partial p_{\parallel}} \left[(\Gamma_1 + \Gamma_2 + \Gamma_3) \frac{\partial \bar{F}}{\partial p_{\parallel}} \right] \quad (3.6.7)$$

with

$$\Gamma_1 = \pi(\mu/\gamma)^2 \int dk_{\parallel} k_{\parallel}^2 S_{zz}^{BB}(k_{\parallel}, k_{\parallel} v_{\parallel}) , \quad (3.6.8)$$

$$\Gamma_2 = \pi q^2 \int dk_{\parallel} S_{zz}^{EE}(k_{\parallel}, k_{\parallel} v_{\parallel}) , \quad (3.6.9)$$

$$\Gamma_3 = 2\pi i (q\mu/\gamma) \int dk_{\parallel} k_{\parallel} S_{zz}^{EB}(k_{\parallel}, k_{\parallel} v_{\parallel}) . \quad (3.6.10)$$

Suppose first that E_z is zero so that only Γ_1 need be considered. Then the action of the magnetic field gradient of these waves upon the magnetic moment of the particles can result in a stochastic acceleration parallel to B_0 much the same as for longitudinal electric fields, but with a different efficiency as a function of wavelength. The factor γ^{-2} makes Γ_1 decrease as p_{\parallel} increases, opposite to what would be needed to make this seem important for the acceleration of relativistic particles. We might imagine p_{\perp} kept comparable to p_{\parallel} with a scattering process--some other waves, for instance--so that μ could be increasing in proportion to γ^2 instead of remaining constant, and we could have $\Gamma_1 \sim \gamma^2$. But this still has the drawback of low over-all efficiency (compared to low-energy particles subject to the same waves) because the Alfvén speed V_A is usually much smaller than that of light, and only a fraction of order V_A/c of an isotropic distribution of relativistic particles would be able to satisfy the condition $v_{\parallel} \sim V_A$ and interact with these waves.

The waves considered here suggest a connection with some recent work of Barnes⁸⁸ and Tidman.⁸⁹ Barnes used the method of SA to find an acceleration of particles by certain magnetohydrodynamic waves (closely related but not identical to our cylindrical waves) and to confirm the damping rate of these waves which he had earlier predicted from the warm-plasma dispersion relation.⁹⁰ What is of most interest here is his point that (nonrelativistic) electrons are better able than ions to respond to a transient push away from a mirror, so that a charge separation and a field E_z will accompany the magnetic fluctuation at the same frequency and wavelength. Barnes estimated that the effects of the electric and magnetic fields would be comparable for thermal particles. Then since μ increases with particle energy while q does not, we have reason to

believe that, for our own interest in suprathermal particles, Γ_1 is the dominant term in (3.6.7).

Tidman also calculated a damping rate for MHD waves, but in his case it is due to gyroresonant, rather than zero-frequency, interaction of waves with a pre-existing suprathermal distribution of particles; thus it really has more to do with Section IV of this chapter than the present section. He estimated that the presence of the cosmic rays would cause appreciable damping of MHD waves in the interstellar medium, but that this was not an important input mechanism for the total energy balance of the particles.

Both of these approaches serve to suggest another condition which should be kept in mind: If some group of waves (of any kind) is to be considered as a source of acceleration of highly suprathermal particles, these waves should be incapable of accelerating the thermal background, i.e., incapable of dumping their energy into unwanted heating instead of carefully "reserving" it all for preferential acceleration of the special class of particles that already have higher than thermal energy. This again suggests that gyroresonant interactions will be better candidates for affecting relativistic particles than zero-frequency interactions, because they have a built-in "selection rule": Waves with low wavenumber and frequency $\omega < \Omega$ will be more or less limited to interacting exclusively with particles for which $\gamma \sim \Omega/\omega$.

We should comment on the discrepancy between (3.6.7) and the results we obtained elsewhere⁸⁶ by the method of SA, although the difference has little effect upon the conclusions drawn in either place. Our present derivation (QL or "enlightened" FP) apparently yields a Fokker-Planck coefficient

$$\left\langle \frac{\Delta p_{\perp}}{\Delta t} \right\rangle = 0, \quad (3.6.11)$$

where we formerly gave a finite result. We believe that the explanation is that the FP approach is not capable (when followed blindly) of recognizing adiabatic invariants when they exist. It is true that the interaction initially tends to decrease p_{\perp} on the average, but because of

the adiabatic invariance this stops as soon as the particle reaches the nearest wave trough rather than continuing indefinitely. This is not so much a real difference between the two methods as it is a fortuitous circumstance that QL should agree with the adiabatic invariant when FP does not, for neither method is strictly applicable; the values of a coordinate controlled by an adiabatic invariant cannot really undergo a random walk such as these methods are intended to describe. There was a difference in that the present approach automatically bypassed the calculation and subsequent neglect of many gyroresonant terms, which were of doubtful consistency with the desired approximations anyhow. This is a concrete example of our remarks in Chapter 2 to the effect that QL both simplifies analysis and avoids errors in comparison with FP.

VII. Spatial Diffusion

Emphasis in the preceding sections has been focused upon the changes in momentum resulting from fluctuating fields, and this will be done again after this section, but we will record a few formulae here showing how drifts in position space naturally arise out of the same theory. (More or less similar results for spatial diffusion have been found by others; see, for example, Moushaan.⁹¹) Considering now the strictly spatial terms on the right-hand side of (3.3.14), we may use (3.3.7b) [in exactly the same way that (3.3.9)-(3.3.13) were used in the remainder of that section] to calculate four real quantities:

$$\begin{aligned}
 D_1 = \text{Re}[\bar{D}_{+-}] = B_o^{-2} \int d^3k \int d\omega \sum_n \left\{ J_n^2 \left[c^2 S_{+-}^{EE} + v_{||}^2 S_{+-}^{BB} + i c v_{||} S_{+-}^{BE} - i c v_{||} S_{+-}^{EB} \right] \right. \\
 \left. + v_{\perp}^2 J_{n+1}^2 S_{zz}^{BB} \right\} \pi \delta(k_{||} v_{||} + n\Omega - \omega) ,
 \end{aligned}
 \tag{3.7.1}$$

$$D_2 = \text{Im}[\bar{D}_{+-}] = B_o^{-2} \int d^3k \int d\omega \sum_n \left\{ J_n^2 \left[-c^2 S_{+-}^{EE} - v_{\parallel}^2 S_{+-}^{BB} - i c v_{\parallel} S_{+-}^{BE} + i c v_{\parallel} S_{+-}^{EB} \right] \right. \\ \left. + v_{\perp}^2 J_{n+1}^2 S_{zz}^{BB} \right\} P \frac{1}{k_{\parallel} v_{\parallel} + n\Omega - \omega} , \quad (3.7.2)$$

$$D_3 = \text{Re}[\bar{D}_{++}] = B_o^{-2} \int d^3k \int d\omega \sum_n \left\{ J_n^2 \left[-\frac{1}{2} c^2 (S_{++}^{EE} + S_{--}^{EE}) \right. \right. \\ \left. + \frac{1}{2} v_{\parallel}^2 (S_{++}^{BB} + S_{--}^{BB}) - \frac{1}{2} i c v_{\parallel} (S_{++}^{BE} + S_{++}^{EB} - S_{--}^{BE} - S_{--}^{EB}) \right] \\ \left. + v_{\perp}^2 J_{n-1} J_{n+1} \cos 2\varphi S_{zz}^{BB} \right\} \pi \delta(k_{\parallel} v_{\parallel} + n\Omega - \omega) , \quad (3.7.3)$$

$$D_4 = \text{Im}[\bar{D}_{++}] = B_o^{-2} \int d^3k \int d\omega \sum_n \left\{ J_n^2 \left[\frac{1}{2} c^2 (S_{++}^{EE} - S_{--}^{EE}) \right. \right. \\ \left. - \frac{1}{2} v_{\parallel}^2 (S_{++}^{BB} - S_{--}^{BB}) + \frac{1}{2} i c v_{\parallel} (S_{++}^{BE} + S_{++}^{EB} + S_{--}^{BE} + S_{--}^{EB}) \right] \\ \left. - v_{\perp}^2 J_{n-1} J_{n+1} i \sin 2\varphi S_{zz}^{BB} \right\} i \pi \delta(k_{\parallel} v_{\parallel} + n\Omega - \omega) . \quad (3.7.4)$$

Again, the arguments are $J_n(k_{\perp} r_g)$ and $S(\vec{k}, \omega)$ throughout. In these terms we will have

$$\left(\frac{D\bar{F}}{Dt} \right)_{xx} = \frac{\partial}{\partial x_g} \left[\frac{D_1 + D_3}{2} \frac{\partial \bar{F}}{\partial x_g} \right] + \frac{\partial}{\partial y_g} \left[\frac{D_1 - D_3}{2} \frac{\partial \bar{F}}{\partial y_g} \right] + \frac{\partial}{\partial x_g} \left[\frac{D_4 - D_2}{2} \frac{\partial \bar{F}}{\partial y_g} \right] + \frac{\partial}{\partial y_g} \left[\frac{D_4 + D_2}{2} \frac{\partial \bar{F}}{\partial x_g} \right] , \quad (3.7.5)$$

which allows anisotropic diffusion. Whenever cylindrically symmetric field spectra are being considered, the ϕ integration will annihilate D_3 and D_4 , and if in addition the spectra are spatially homogeneous (3.7.5) will reduce to the simple form

$$\left(\frac{D\bar{F}}{Dt}\right)_{xx} = \frac{1}{2} D_1 \nabla_{\perp}^2 \bar{F} .$$

In the case of electric fields alone (as in Section IV), the limit for $k_{\perp} r_g \ll 1$ of the important coefficient D_1 becomes

$$D_1 \rightarrow \frac{\pi c^2}{B_0^2} \int d^3 k S_{+-}^{EE}(\vec{k}, k_{\parallel} v_{\parallel}) . \quad (3.7.6)$$

Notice that it is determined by a different part of the spectrum than is $\bar{\Gamma}_{tt}$ in (3.4.3); this spatial diffusion is just the result of $\Delta \vec{E} \times \vec{B}_0$ drift. In the case of magnetic fields (Section V),

$$D_1 \rightarrow \frac{\pi}{B_0^2} \int d^3 k \left[v_{\parallel}^2 S_{+-}^{BB}(\vec{k}, k_{\parallel} v_{\parallel}) + v_{\perp}^2 S_{zz}^{BB}(\vec{k}, k_{\parallel} v_{\parallel} - \Omega) \right] ; \quad (3.7.7)$$

gyroresonant pumping and the following of displaced field lines both contribute here to spatial diffusion. The transport of particles by this method in a plasma of very low collision frequency can greatly enhance such transport coefficients as viscosity and heat conductivity in directions perpendicular to the magnetic field. Tsuda has considered the application of these ideas to some problems of momentum and energy transport in the Earth's magnetosphere and ionosphere.⁹²

Finally, (3.3.14) contains mixed terms involving both position and momentum derivatives. They could be written out in the same way as were (3.3.22) and (3.7.5) and their coefficients, but we shall not do so here. In the cylindrically symmetric case which is of most interest to us these

terms all vanish, leaving diffusions in position and in momentum which are independent of one another.

VIII. Nonuniform Conditions

The relatively simple idealized cases for the choice of steady fields G^μ have already been exhausted by our consideration in Section II of $\vec{E}_0 = \vec{B}_0 = 0$ and in Section III of $\vec{E}_0 = 0$, $\vec{B}_0 = \text{constant}$. It is not within the scope of this work to treat further cases in any detail, but we may comment briefly on what such a study might involve.

There are two other cases of constant and uniform fields possible: $\vec{E}_0 = \text{constant}$, $\vec{B}_0 = 0$, and \vec{E}_0 and \vec{B}_0 both constant and parallel to one another. These cases would both involve ordered runaway acceleration^{93,94} and its chain of consequences, and before undertaking the task of calculating statistical effects in their presence, one would consider carefully whether the fluctuation phenomena are likely to have any important role in comparison to the runaway. The transition from linear to parabolic type unperturbed orbits would make the evaluation of Fourier integrals much more difficult, probably requiring the use of Airy functions or something comparable.

The difficulties of considering nonuniform background fields will be evident to one who has followed the mathematics of Section III and Appendix C in detail. Equation (2.4.5) certainly suggests that adiabatic invariants and their conjugate phase variables would be used as coordinates whenever they exist. Birmingham, Northrop, and Fälthammar have been working on such a theory for particles in the Earth's magnetic field, which will refine present ideas about stochastic acceleration by violation of the so-called third invariant in certain geomagnetic disturbances.⁹⁵

Chapter 4

SYNCHROTRON RADIATION

I. The Emitted Radiation

A charged particle moving in a magnetic field must, as a result of its continual centripetal acceleration, radiate energy in the form of electromagnetic waves. This is commonly called cyclotron radiation (for nonrelativistic particles), synchrotron radiation (relativistic), or (usually in the Russian literature) magnetobremssstrahlung; and it must be taken into account if the possibility of stochastic acceleration in astrophysics is to be considered. The basic theory of this radiation has been established for over fifty years,⁹⁶ and it has recently been the subject of a thorough review from the point of view of astrophysical applications.⁹⁷ We shall merely note in this section a few of its principal properties which will be of use to us later, as given in the review mentioned or in a typical textbook.⁹⁸

The total power radiated by a particle of mass m , charge q , and total energy $W = \gamma mc^2$, moving with pitch angle θ , in a uniform magnetic field of strength B , is

$$P = \frac{2q^4 B^2}{3m^2 c} \sin^2 \theta (\gamma^2 - 1). \quad (4.1.1)$$

The last factor is equivalent to $\beta^2 \gamma^2$, showing how this power becomes small for nonrelativistic motion; in that case the radiation is concentrated at the cyclotron frequency $\Omega_0 = qB/mc$ and is of dipole type.

But we shall be interested almost exclusively in the ultrarelativistic case, for which many harmonics come into play and give a practically continuous spectrum extending up to frequencies of order

$$\omega_c = \sin \theta \gamma^2 \Omega_0.$$

This spectrum for single-particle radiation varies approximately as $\omega^{1/3}$ below ω_c and as $\exp(-2\omega/3\omega_c)$ above, as pictured in Jackson's Fig. 14.11 (Ref. 98, p. 487). In spatial distribution, this radiation falls off exponentially for angles farther away from the cone of directions traced out by the velocity vector than

$$\theta_c = \frac{1}{\gamma} \sqrt{\frac{\omega_c}{\omega}}.$$

The radiation has in general an elliptical polarization, with the principal electric vector at right angles to the projection of the magnetic field on a plane perpendicular to the observer's line of sight, as illustrated in Ginzburg and Syrovatskii's Fig. 5 (Ref. 97, p. 308). The degree of polarization averaged over all emission angles increases from 1/2 at low frequencies to 1 at high frequencies; further details are abundant in Reference 97.

Figures 3 and 4 present graphs which we have found convenient for quick reference on the properties of single-particle synchrotron radiation. Using W and $B_{\perp} = B \sin \theta$ for axes (logarithmically scaled), we can plot sets of straight lines representing constant values of other quantities. Lines of slope +1 may be labeled with either the gyrofrequency $\Omega = \Omega_0/\gamma$ or the gyroradius $r_g = c/\Omega$ and those of slope -1/2 represent the characteristic frequency

$$\nu_m = 0.43(\omega_c/2\pi)$$

at which the greatest radiation occurs. Lines of constant total power (slope -1) are omitted in favor of those with slope -2 representing the characteristic time $\tau = W/P$ in which the particle would lose one-half of its original energy. Although in some kinematic respects (such as Ω or r_g) electrons and protons have similar properties when both are ultra-relativistic, this is decidedly untrue of their radiation. For a given energy the radiated power P still depends upon the rest mass as m^{-4} , τ as m^4 , and ω_c as m^{-3} ; this is the difference between the two graphs. There are two extra cautionary lines on each graph above which

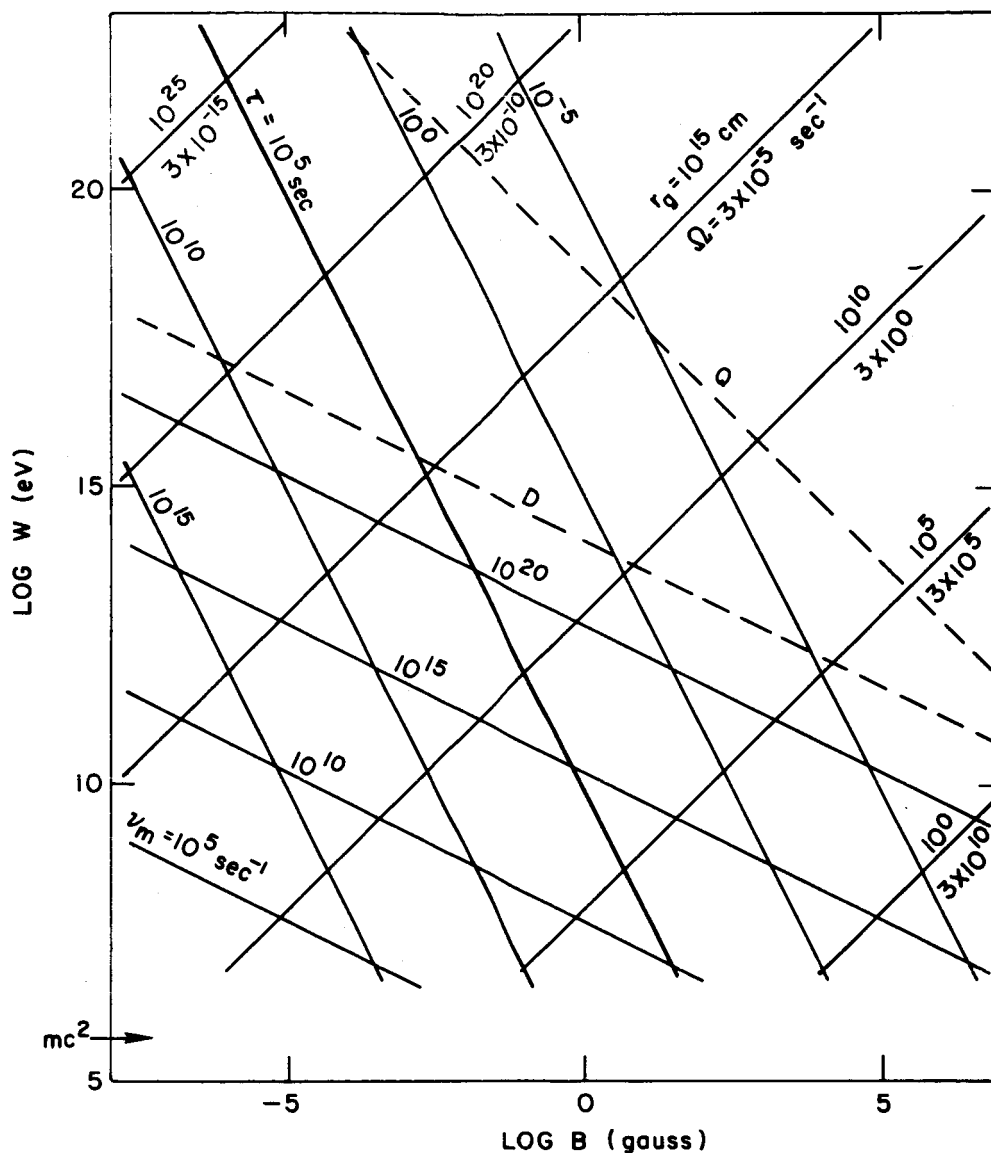


FIG. 3. SYNCHROTRON RADIATION BY ELECTRONS.

this information should be regarded only as a rough estimate: at line D the radiation damping is becoming important in determining the particle's motion, and at line Q the energy of a single photon at the most likely frequency exceeds the particle's original energy, so it must already have become important before this to use quantum mechanics instead of classical radiation theory. We shall not become involved in this last question, but an interesting review of its possibilities has recently appeared.⁹⁹

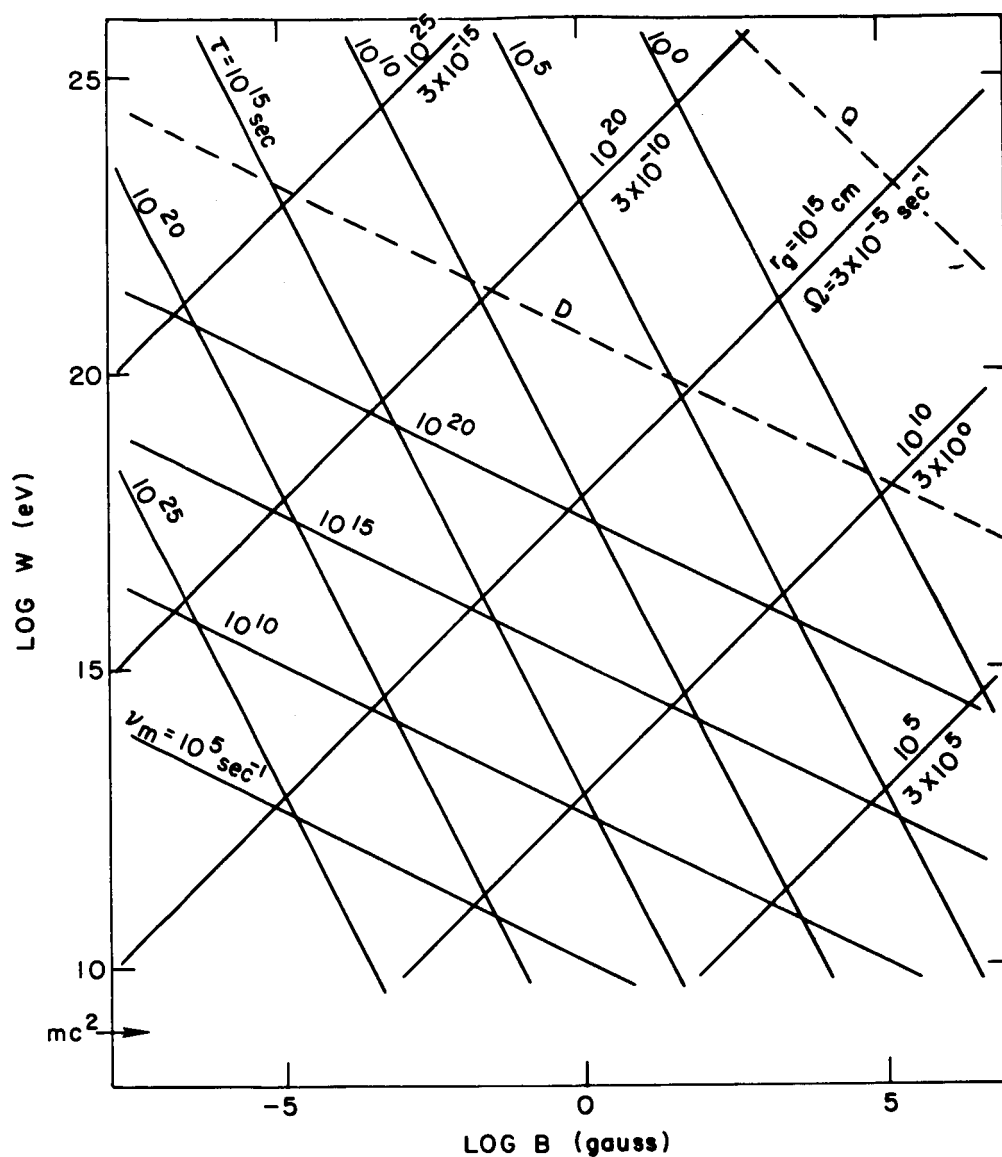


FIG. 4. SYNCHROTRON RADIATION BY PROTONS.

On the assumption that there is no process locking the phases of different particles together and causing them to radiate coherently, the radiation from a large number of particles with different energies may be found simply by integrating their individual intensities. This has the well-known result that a distribution $N(W) = K_1 W^{-n}$ of particle energies in a source will produce an observed radiation spectrum which is described by a power law also:

$$I(\nu) \propto \nu^{-s}, \quad s = \frac{1}{2}(n-1), \quad \text{if } n > \frac{1}{3}. \quad (4.1.2)$$

Each frequency ν is contributed mainly by those particles for which $\nu_m \approx \nu$. The integrated polarization becomes linear, in the direction already described, with strength

$$\Pi = \frac{n+1}{n+7/3} \quad (4.1.3)$$

which varies from $1/2$ to 1 as n increases from $1/3$ to ∞ .

If this is further integrated along a line of sight where the direction of the magnetic field is not everywhere the same, the degree of polarization will be reduced, going of course to zero in the limit where all directions of field are equally likely. If $N(W) = K_1 W^{-n}$ continues to hold for various pitch angles and positions in space, then (4.1.2) remains true; any variation of K_1 will not matter unless there is such a great anisotropy that the distribution function changes appreciably over a very small angle of the order of γ^{-1} .

As described above, a synchrotron spectrum can never increase with frequency faster than $\nu^{1/3}$, but there are processes that can alter this. Synchrotron radiation can be self-absorbed if it passes through a region containing other energetic particles; for optically thick sources this can result in $I(\nu) \propto \nu^{5/2}$ at low frequencies.⁹⁷ Another complication is the refractive index of the ambient plasma, which has a characteristic frequency $\nu_p = (ne^2/\pi m)^{1/2}$; for frequencies $\nu \lesssim \nu_p$ this will influence the radiation process. Hornby and Williams¹⁰⁰ have considered these and other possible explanations of observed low-frequency cutoffs in the radio spectra of several extragalactic radio sources.

II. Effect of Radiation Loss on Particles

More attention has generally been given to the radiation itself than to its reaction upon the radiating particles, but this too is a straightforward problem soluble in principle, and it is of prime interest to us here. Dirac's theory of relativistic classical point electrons has (see Ref. 98, p. 609) the equation of motion

$$\frac{dp_{\mu}}{d\tau} = F_{\mu}^{\text{ext}} + F_{\mu}^{\text{rad}} \quad (4.2.1)$$

under the external forces F_{μ}^{ext} , where the radiative reaction is given by

$$F_{\mu}^{\text{rad}} = \frac{2q^2}{3mc^3} \left[\frac{d^2 p_{\mu}}{d\tau^2} - \frac{p_{\mu}}{m^2 c^2} \left(\frac{dp_{\nu}}{d\tau} \frac{dp_{\nu}}{d\tau} \right) \right]. \quad (4.2.2)$$

Here p_{μ} is the particle's four-momentum (W, \vec{p}) , τ its proper time, and we wish to calculate F_{μ}^{rad} as a small self-generated correction to F_{μ}^{ext} . Then we may use the unperturbed solution

$$p_{\mu} = (W, p_{\perp} \cos \Omega t, p_{\perp} \sin \Omega t, p_{\parallel}) \quad (4.2.3)$$

and $dt = \gamma d\tau$ to calculate in the frame of observation that

$$\frac{dp_{\perp}^{\text{rad}}}{d\tau} = - \frac{2q^4 B^2}{3m^3 c^5} \left[1 + \frac{p_{\perp}^2}{m^2 c^2} \right] \frac{p_{\perp}}{\gamma}, \quad (4.2.4)$$

and

$$\frac{dp_{\parallel}^{\text{rad}}}{dt} = - \frac{2q^4 B^2}{3m^3 c^5} \frac{p_{\perp}^2}{m^2 c^2} \frac{p_{\parallel}}{\gamma}. \quad (4.2.5)$$

The fourth component, dW^{rad}/dt , reproduces (4.1.1).

Let us define

$$C = \frac{2q^4}{3m^3 c^5}; \quad (4.2.6)$$

for electrons this is equal to $1.8 \times 10^{-9} \text{ sec}^{-1} \text{ gauss}^{-2}$. Then we may either write the effect of the radiation damping upon the distribution function

$$\left(\frac{\partial F}{\partial t}\right)_{\text{rad}} = \frac{CB^2}{p_{\perp}} \frac{\partial}{\partial p_{\perp}} \left[\left(1 + \frac{p_{\perp}^2}{m^2 c^2}\right) \frac{p_{\perp}^2 F}{\gamma} \right] + CB^2 \frac{\partial}{\partial p_{\parallel}} \left[\frac{p_{\perp}^2}{m^2 c^2} \frac{p_{\parallel} F}{\gamma} \right] \quad (4.2.7)$$

or transform to total momentum and pitch angle to obtain

$$\left(\frac{\partial F}{\partial t}\right)_{\text{rad}} = \frac{CB^2 \sin^2 \theta}{p^2} \frac{\partial}{\partial p} [\gamma p^3 F] + \frac{CB^2}{\gamma \sin \theta} \frac{\partial}{\partial \theta} [\sin^2 \theta \cos \theta F] \quad (4.2.8)$$

A set of stationary, separable solutions to this equation can easily be written:

$$F \sim \frac{1}{\gamma p^3 \sin^2 \theta \cos \theta} \left| \frac{p \cos \theta}{\gamma} \right|^x \quad (4.2.9)$$

Locally, these are valid for any value of x , but from the global viewpoint they are somewhat pathological functions: First, they are not normalizable, always giving a logarithmic divergence at $\theta = 0$ regardless of the value of x , but this might be expected physically since the particle pitch angles are always being decreased by the radiation. Second, to avoid a nonphysical divergence at $\theta = \pi/2$ we would require $x > 0$, and if an n^{th} derivative is to be defined at $\theta = \pi/2$ this would be further limited to $x = 1, 2, \dots, n$ or $x > n-1$. Finally, there must be a source of particles at $p = \infty$ to support such a solution everywhere.

Our interest will be primarily in the ultrarelativistic limit where (4.2.8) reduces to

$$\left(\frac{\partial F}{\partial t}\right)_{\text{rad}} = \frac{CB^2 \sin^2 \theta}{mc p^2} \frac{\partial}{\partial p} (p^4 F) [1 + O(\gamma^{-2})] \quad (4.2.10)$$

which shows that in this limit it is a good approximation to assume that all particles merely lose energy according to (4.1.1) without any change in their pitch angles. A stationary solution, from either (4.2.9) or (4.2.10), would be $F \sim p^{-4}$ (times any reasonable function of θ). This would correspond to $n = 2$, $s = 1/2$ in (4.1.2), but we should not hastily conclude that observed spectral indices will exhibit any tendency to have the value 0.5.

III. Dynamic Radiation Spectra

The general solution of (4.2.10) can in fact be written explicitly, since it contains only first derivatives; the quantity $p^4 F$ remains constant along the characteristics

$$\frac{dp}{dt} = -Lp^2, \quad \frac{d\theta}{dt} = 0, \quad (4.3.1)$$

where

$$L = \frac{CB^2 \sin^2 \theta}{mc}, \quad (4.3.2)$$

so that an initial distribution $F_o(p, \theta, 0)$ will become at any later time

$$\begin{aligned} F(p, \theta, t) &= (1 - Lpt)^{-4} F_o\left(\frac{p}{1 - Lpt}, \theta, 0\right) & (pt < L^{-1}) \\ &= 0 & (pt > L^{-1}) . \end{aligned} \quad (4.3.3)$$

Of particular interest would be the class of particle spectra which can be written as the product of one function of p alone with another function of pt alone, and this is just the class of initially power-law distributions. Using n in the same sense as in (4.1.2), these are

$$F_n(p, \theta, t) = A(\theta) (1 - Lpt)^{n-2} p^{-n-2} \quad (pt < L^{-1}) . \quad (4.3.4)$$

For a given t and reasonable values of n , the original spectrum is for practical purposes unchanged except within the last decade of p before the cutoff at $1/Lt$. Then the radiation spectrum from particles of a given pitch angle will also be stationary and as given by (4.1.2) up to within the last decade before

$$\nu_c(\theta, t) = \frac{qB \sin \theta \left(\frac{1}{Lt}\right)^2}{2\pi m c} = \frac{9m^5 c^9}{8\pi q t^7 B^3 \sin^3 \theta} = \frac{2.8 \times 10^{23}}{t^2 B^3 \sin^3 \theta} \quad \text{for electrons.} \quad (4.3.5)$$

Above $\nu_c(\theta, t)$ the spectrum will fall off exponentially, and just below its behavior will depend on n ; qualitatively it is a smoothed replica of the particle spectrum (4.3.2), as shown in Fig. 5, and quantitatively there are available exact formulae (such as Ref. 97, equation 3.20) from which these falloffs could be calculated accurately, but that is not important to us here.

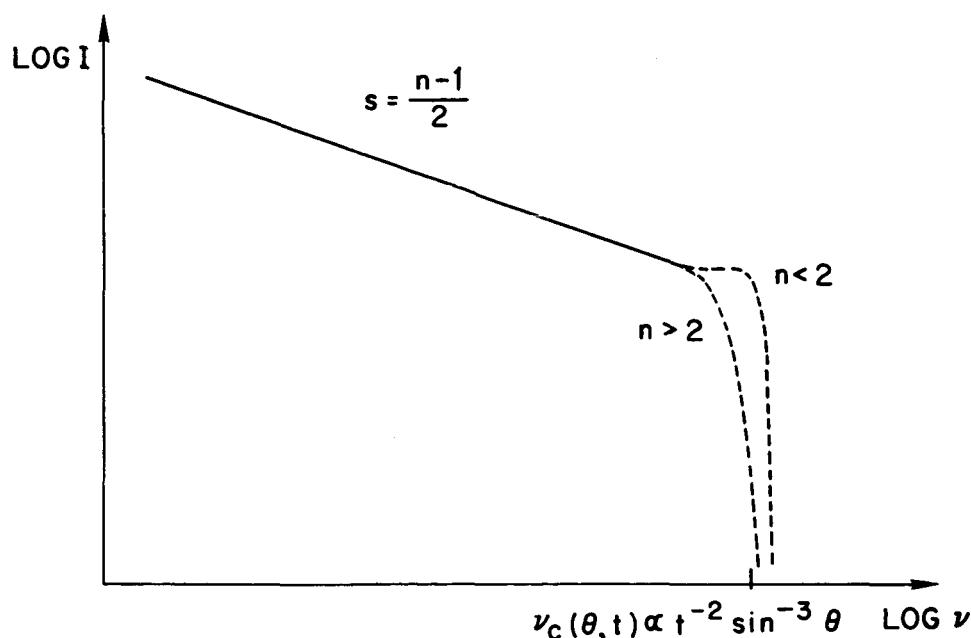


FIG. 5. RADIATION SPECTRUM FROM UNDISTURBED PARTICLES AT A SINGLE PITCH ANGLE.

We are now in a position to show an interesting result, apparently due to Kardashev,⁶⁷ which has been used recently by Kellermann¹⁰¹ to

propose an explanation for the distribution of spectral indices of observed radio sources. In neither of these places was it clear to our satisfaction just how this result was found, or what were the conditions for its validity, so its derivation will be given briefly here. We must assume (i) that an isotropic power-law spectrum of particles is injected uniformly throughout a certain region of space at the time $t = 0$; (ii) that the strength B of the magnetic field is nearly constant in this region, yet at the same time that the direction of this field changes sufficiently from one part to another so that the radiation received by an external observer will constitute a sampling taken equally from all pitch angles; and (iii) that there is no other process taking place which could significantly change the pitch angle for any of the particles over any time scale to be discussed here. We shall not ask whether these rather stringent conditions are met in any real object until a later chapter; for now we only ask what their consequences would be.

The spectrum seen by the observer under these conditions will be

$$I(\nu, t) = \int_0^\infty p^2 dp \int_0^\pi \sin \theta d\theta F_n(p, \theta, t) H(p, \theta, \nu), \quad (4.3.6)$$

where F_n is given by (4.3.4) with A independent of θ , and

$$H(p, \theta, \nu) \propto B \sin \theta J(\nu/\nu_c), \quad \nu_c(\theta, p) = \frac{q}{2\pi m^3 c^3} B p^2 \sin \theta, \quad (4.3.7)$$

describes the intensity contributed by a single particle at p, θ . $J(x)$ is a known function proportional to $x^{1/3}$ and $\exp(-2x/3)$ for small and large x respectively, as described in the first section of this chapter. Thus

$$I(\nu, t) \propto B \int_0^\pi d\theta \int_0^{1/Lt} dp \sin^2 \theta p^{-n} (1 - Lpt)^{n-2} J(\nu/\nu_c), \quad (4.3.8)$$

where the angular integral must be written first because the momentum cutoff $1/Lt$ depends on θ . Now we take advantage of the nature of J by making $x = v/v_c$ instead of p the independent variable for the second integration:

$$I(v, t) \propto B \int_0^\pi d\theta \sin^2 \theta \int_{v/v_c(\theta, t)}^\infty \frac{dx}{2x} [p(x; \theta, v, B)]^{1-n} (1 - Lpt)^{n-2} J(x) \quad (4.3.9)$$

or

$$I(v, t) \propto B^{\frac{1}{2}(n+1)} v^{-\frac{1}{2}(n-1)} \int_0^\pi d\theta \sin^{\frac{1}{2}(n+3)\theta} \int_{v/v_c(\theta, t)}^\infty dx J(x) x^{\frac{1}{2}(n-3)} \left[1 - \sqrt{\frac{v}{x v_c(\theta, t)}} \right]^{n-2} . \quad (4.3.10)$$

Now (4.3.5) states that

$$v_c(\theta, t) = v_c\left(\frac{1}{2} \pi, t\right) / \sin^3 \theta \geq v_c\left(\frac{1}{2} \pi, t\right) , \quad (4.3.11)$$

so for small frequencies the lower limit of the x integration is always much less than one, and in fact it can be approximately set to zero for evaluation of the factors $J(x) x^{\frac{1}{2}(n-3)}$ if $[1/3 + (n-3)/2] > -1$, or $n > 1/3$. In order that the final factor may be approximated by unity we must have $n > 1$, and this makes it possible to write

$$v \ll v_c\left(\frac{1}{2} \pi, t\right):$$

$$I(v, t) \propto B^{\frac{1}{2}(n+1)} v^{-\frac{1}{2}(n-1)} \int_0^\pi d\theta \sin^{\frac{1}{2}(n+3)\theta} \int_0^\infty dx J(x) x^{\frac{1}{2}(n-3)} , \quad (4.3.12)$$

where the integrals are now functions of n only--the entire dependence on time (none) and frequency is explicitly shown. As could easily have been guessed, this is no different than the low-frequency part of Fig. 5.

But for high frequencies there are a few particles with sufficiently small pitch angles that the exponential cutoff has not yet affected their radiation. This may best be calculated by interchanging the order of integration:

$$\int_0^{\frac{1}{2}\pi} d\theta \int_{\nu \sin^3 \theta / \nu_c}^{\infty} dx = \int_0^{\infty} dx \int_0^{\min[\frac{1}{2}\pi, \arcsin(x\nu_c/\nu)^{1/3}]} d\theta . \quad (4.3.13)$$

Since only values $x \lesssim 1$ are important, for high frequencies the second upper limit on θ will prevail. In order to bring the dependence on ν out of the limit of integration we define $z = (\nu/\nu_c x)^{1/3} \sin \theta$, which finally leads to

$$\nu \gg \nu_c \left(\frac{1}{2} \pi, t \right):$$

$$I(\nu, t) \propto B^{(n+1)/2} \nu^{-(n-1)/2} \left[\frac{\nu_c \left(\frac{1}{2} \pi, t \right)}{\nu} \right]^{\frac{n+5}{6}} \int_0^{\infty} dx J(x) x^{2(n-1)/3} \int_0^1 dz z^{(n+3)/2} (1-z^{3/2})^{n-2} . \quad (4.3.14)$$

This approximation also requires $n > 1$, actually for the same reason, so that the z integration will converge. The result, as sketched in Fig. 6, is again a power-law radiation spectrum, but with an index

$$s_{\text{high}} = \frac{4}{3} s_{\text{low}} + 1 \quad (4.3.15)$$

and an intensity decreasing in time as

$$I_{\text{high}} \propto t^{-\frac{n+5}{3}} , \quad (4.3.16)$$

which must be faster than t^{-2} . These are the results stated by Kardashev.⁶⁷ Again, we have equations with which the exact shape of the kink could be calculated if there were need for it.

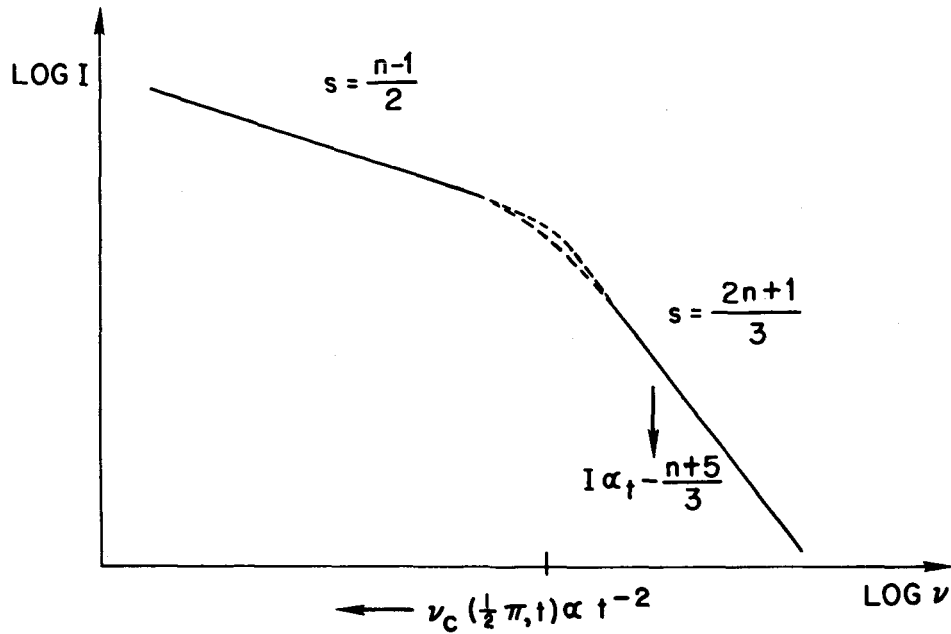


FIG. 6. THE SPECTRUM OF FIG. 4 AVERAGED OVER AN ISOTROPIC DISTRIBUTION OF RADIATING PARTICLES.

IV. Continuous Injection of Energetic Particles

There is one final extension of these results which involves no other processes in an explicit way. Suppose that, rather than a single burst of energetic particles as studied in Section III, there is a continuous injection of energetic particles so that (4.2.10) is generalized to

$$\frac{\partial F}{\partial t} = \frac{L(B, \theta)}{p^2} \frac{\partial}{\partial p} (p^4 F) + Q(p, \theta, t). \quad (4.4.1)$$

The source function Q must represent "sudden" injection, either from a different region of space, or from a fast acceleration process, or from the sudden appearance of new particles from nuclear reactions, etc.; the point is that this does not include any continuous acceleration process which proceeds so slowly that it must compete with radiation loss to get the particles up to the energy of "injection." The latter case will be dealt with in the next chapter.

A general solution to (4.4.1) may be written by integrating (4.3.3) over all times previous to t . It is convenient to change the origin so that $t = 0$ now stands for the present time, and we obtain

$$F(p, \theta, 0) = \int_{-\frac{1}{Lp}}^0 dt (1 + Lpt)^{-4} Q\left(\frac{p}{1 + Lpt}, \theta, t\right). \quad (4.4.2)$$

In particular, if Q is constant in time the variable of integration may be changed from t to $p' = p/(1 + Lpt)$, which gives

$$F(p, \theta) = \frac{1}{Lp^4} \int_p^\infty dp' p'^2 Q(p', \theta). \quad (4.4.3)$$

This would also have been obtained directly by asking for a stationary solution of (4.4.1).

Finally, if we should have a power-law $Q = A(\theta)p^{-(n_o+2)}$ this would become

$$F(p, \theta) \propto \frac{A(\theta) p^{-(n_o+3)}}{B^2 \sin^2 \theta} \quad (n_o > 1), \quad (4.4.4)$$

indicating that the radiation loss steepens the injected spectrum by one power. Since there is no cutoff involved here, the dependence on θ does not matter and the radiation spectrum is given by

$$I(\nu) \propto B^{\frac{1}{2}(n_o-2)} \nu^{-\frac{1}{2}n_o}. \quad (4.4.5)$$

For $n_o < 1$ it is necessary that Q have a cutoff at some p_{\max} ; then we would have $F(p, \theta) \sim p^{-4}$ up to a sharp cutoff at p_{\max} , and $I(\nu) \sim B^{-\frac{1}{2}} \nu^{-\frac{1}{2}}$ at low frequencies with a turnover somewhat like that in Fig. 6. This would depend on $A(\theta)$ now, but the subject will not be pursued here since values of n_o this low are unlikely to concern us anyhow.

Equation (4.4.5) is involved along with (4.3.12) in the Kellermann proposal,¹⁰¹ and (4.4.3) has been used by followers of Alfven to discuss radiation from the electron debris of proton-antiproton annihilation.¹⁰²

Chapter 5

MODELS OF COMBINED EFFECTS

I. Introduction

We should clearly state here a point which will be basic for this and the following chapter: The phenomena involved in this study extend over a wide range, usually a number of decades, in such independent variables as particle energy and radiation frequency. When these variables are considered on logarithmic scales, even a complicated equation describing many effects will usually reduce in certain subranges to a balance between two dominant terms. Insofar as we can consider effects one pair at a time and avoid the problem of joining these regions, the analysis will certainly be simpler. But these transitions from one region or pair of terms to another are ordinarily smooth, and take place within roughly one decade, so that an attempt to get very general solutions including three or four effects at once will seldom be justified by anything that can be learned from them. That is, observational data are usually not of sufficient accuracy for meaningful detailed comparison with any prediction within such a small range.

We shall also sometimes make use of "model terms." This means that a term known to describe an effect correctly may sometimes prove inconvenient to use, and we may substitute for it some other expression which is more tractable for solution, but which we have reason to believe will retain those essential characteristics of the original expression that are most important to the final result.

II. Scattering and Radiation

As the first example, consider the effect on a spatially uniform particle distribution of simultaneous radiation loss (in an average field B) and scattering (by magnetic fluctuations or otherwise). Rather than to study transient effects, it is easier to assume a continuous isotropic power-law injection of particles and then to study the steady-state solution of the equation

$$\frac{\partial F}{\partial t} = \frac{1}{p^2 \sin \theta} \frac{\partial}{\partial \theta} \left[\sin \theta \Gamma_{\theta\theta} \frac{\partial F}{\partial \theta} \right] + \frac{CB^2}{mc} \frac{\sin^2 \theta}{p^2} \frac{\partial}{\partial p} [p^4 F] + A p^{-(n_o+2)} . \quad (5.2.1)$$

Here C is the constant given by (4.2.6), and A is the same quantity appearing in (4.4.4) but is now being assumed constant. $\Gamma_{\theta\theta}$, given by (3.3.17), may depend on θ and (perhaps only weakly) on p , but it will be possible to draw some conclusions by only using a typical magnitude of $\Gamma_{\theta\theta}$ as if it were constant. (The notation \bar{F} , $\bar{\Gamma}$ served its purpose in Chapter 3, and the bars will not be used below.)

Equation (4.1.1) shows that the power radiated by a particle with parameters p and θ will be appreciably altered by changes of order $\Delta p \sim p$ and $\Delta \theta \sim \sin \theta$, and an estimate must be made as to which of these changes is more likely to take place. But from (5.2.1) we may write

$$\frac{1}{T_{\text{scatt}}} \sim \frac{\Gamma_{\theta\theta}}{p^2} \frac{1}{(\Delta \theta)^2} , \quad \frac{1}{T_{\text{rad}}} \sim \frac{CB^2}{mc} \frac{p^2 \sin^2 \theta}{\Delta p} , \quad (5.2.2)$$

so these terms will be of comparable importance along an "influence boundary" given by

$$\frac{\Gamma_{\theta\theta}}{p^2 \sin^2 \theta} \approx \frac{CB^2 p^2 \sin^2 \theta}{mcp} \equiv K p \sin^2 \theta \quad (5.2.3)$$

or

$$p^3 \sin^4 \theta \approx \Gamma_{\theta\theta} / K \equiv p_0^3 . \quad (5.2.4)$$

We have indicated in Fig. 7 how the radiation dominates the particle trajectories in phase space for sufficiently large p , where the solution

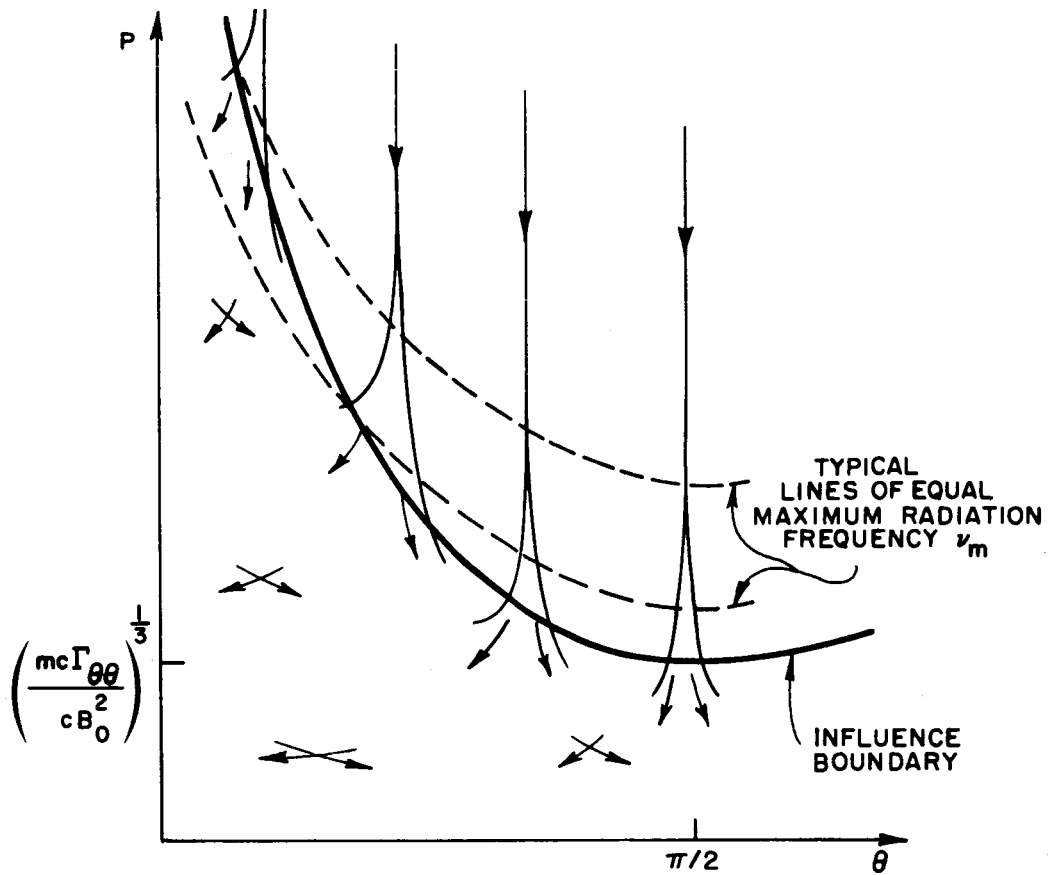


FIG. 7. SCHEMATIC PARTICLE TRAJECTORIES WITH RADIATION AND SCATTERING.

must be given by (4.4.4), and how scattering will keep the distribution function isotropic for small p even though the particles continue to radiate with an effective value

$$\int_0^{\pi/2} d\theta \sin^3 \theta = \frac{2}{3}$$

replacing $\sin^2 \theta$ in (4.1.1).

A similar conclusion may be drawn more rigorously by applying the operator

$$\frac{1}{2} \int_0^\pi d\theta p^2 \sin \theta$$

to (5.2.1) (with $\partial F / \partial t = 0$). The presence of any nonzero $\Gamma_{\theta\theta}$ prevents singularities in F as a function of θ , so there is no question that $\sin \theta \Gamma_{\theta\theta} (\partial F / \partial \theta) \rightarrow 0$ at $\theta = 0, \pi$: the remaining expression is completely independent of $\Gamma_{\theta\theta}$:

$$\frac{\partial}{\partial p} \left[p^4 \frac{1}{2} \int_0^\pi d\theta \sin^3 \theta F(p, \theta) \right] + \frac{A}{K} p^{-n_0} = 0. \quad (5.2.5)$$

For $n_0 > 1$ we may integrate over p and obtain

$$\frac{1}{2} \int_0^\pi d\theta \sin^3 \theta F(p, \theta) = \frac{A}{K} \frac{p^{-(n_0+3)}}{n_0-1}, \quad (5.2.6)$$

which is clearly a generalization of (4.4.4). From this it may be seen that wherever F is isotropic (as for $p \ll p_0$) its dependence on p is determined as well, and in fact the replacement $\sin^2 \theta \rightarrow 2/3$ is also correct in (4.4.4). This provides sufficient information to investigate the spectrum of radiation which would come from this distribution.

Figure 7 also indicates how the frequencies of maximum radiation of the particles near the influence boundary are predominantly quite close to the value

$$\nu_0 = \nu_m \left(p_0, \frac{1}{2} \pi \right) = \frac{0.43}{2\pi} \frac{qB^{-1/3}}{(mc)^{7/3}} \left(\frac{\Gamma_{\theta\theta}}{C} \right)^{2/3}. \quad (5.2.7)$$

The radiation at high frequencies ($\nu \gg \nu_0$) will be determined almost entirely by particles with $p^3 \sin^4 \theta \gg p_0^3$, for which

$$F_{\text{high}}(p, \theta) = \frac{A}{(n_o - 1)K} \frac{p^{-(n_o + 3)}}{\sin^2 \theta}, \quad (5.2.8)$$

and the radiation at $\nu \ll \nu_o$ will come mainly from particles with $p \ll p_o$, for which

$$F_{\text{low}}(p, \theta) = \frac{A}{(n_o - 1)K} \frac{p^{-(n_o + 3)}}{2/3}. \quad (5.2.9)$$

These may be used in (4.3.6), with the same transformations which led before to (4.3.12), to find that

$$I_{\text{high}}(\nu) = c_1 \nu^{-\frac{1}{2}n_o}, \quad I_{\text{low}}(\nu) = c_2 \nu^{-\frac{1}{2}n_o}, \quad (5.2.10)$$

where the ratio of the two constant coefficients may readily be calculated:

$$\frac{c_1}{c_2} = \frac{\int_0^\pi d\theta \sin^{\frac{1}{2}n_o} \theta}{\frac{3}{2} \int_0^\pi d\theta \sin^{\frac{1}{2}(n_o + 4)} \theta} = \frac{2(n_o + 4)}{3(n_o + 2)}. \quad (5.2.11)$$

As n_o varies from its minimum allowed value of one to a maximum of infinity, this ratio varies only from 10/9 to 2/3, and in fact the most reasonable values of n (perhaps 1.5 to 2.5) give values for this ratio (22/21 to 26/27) which are quite close to unity. We see no reason to expect anything other than smooth joining of these nearly identical solutions in the region around ν_o , so we must conclude that the addition of scattering in pitch angle cannot be expected to cause any

noticeable feature in the radiation spectrum of particles being continuously injected according to an isotropic power law (nor, probably, for any other reasonably similar injection spectrum).

III. Stochastic Acceleration and Radiation

We proceed to consider the behavior of the distribution function under the influence of statistical acceleration (nonzero Γ_{tt} and Γ_{zz} , as indicated qualitatively in Chapter 3, Section IV), but now with radiation loss also being taken into account. It proves unsatisfactory to begin straightforwardly with terms from (3.3.22) and (4.2.10), for the radiation and diffusion terms are in different coordinate systems and are not related to one another in form in such a way as to make the combination readily integrable. Instead, tractable "model terms" may be formed by supposing that strong scattering is present also and assuming that this keeps the distribution isotropic. Then we may use (D.5) and (4.2.10), operate with

$$\frac{1}{2} \int_0^\pi d\theta \sin \theta ,$$

and obtain

$$\frac{\partial F}{\partial t} = \frac{2K}{3p^2} \frac{\partial}{\partial p} [p^4 F] + \frac{1}{p^2} \frac{\partial}{\partial p} \left[p^2 D_{\text{eff}} \frac{\partial F}{\partial p} \right] , \quad (5.3.1)$$

where the effective value of D_{pp} (from Appendix D) is given by

$$D_{\text{eff}} = \frac{1}{2} \int_0^\pi d\theta \left[\sin^3 \theta \Gamma_{tt} + \sin \theta \cos^2 \theta \Gamma_{zz} \right] . \quad (5.3.2)$$

If Γ_{tt} and Γ_{zz} are isotropic, D_{eff} becomes $2/3 \Gamma_{tt} + 1/3 \Gamma_{zz}$.

In order to investigate stationary solutions, we set $\partial F / \partial t = 0$, whereupon one integration may be immediately performed:

$$\frac{2}{3} K p^4 F + p^2 D_{\text{eff}} \frac{\partial F}{\partial p} = \text{constant} . \quad (5.3.3)$$

For a physically acceptable D_{eff} which is strongly cut off above some (perhaps very high, but finite) energy, F must also approach zero very rapidly for $p \rightarrow \infty$ and the constant in (5.3.3) must be zero. Then the second integration is also easily done, yielding

$$F(p) = F(0) \exp \left[- \frac{2}{3} K \int_0^p \frac{p'^2 dp'}{D_{\text{eff}}(p')} \right] . \quad (5.3.4)$$

The significance of this expression may be seen from the following examples. First, in any region of p where D_{eff} is constant,

$$F(p) \sim \exp \left[- (p/p_0)^3 \right] ;$$

this represents roughly a steepening by the radiation of the spectrum (3.4.1). Second, for Fermi's acceleration mechanism $D_{\text{eff}} \sim p$ and

$$F(p) \sim \exp \left[- (p/p_0)^2 \right] .$$

In both of these cases there is a characteristic momentum p_0 [equal to $(9D/2K)^{1/3}$ and $(3D/pK)^{1/2}$, respectively] which plays the role of an influence boundary between diffusion and radiation terms, or a barrier beyond which particles cannot readily diffuse. Likewise in both these cases the radiation spectrum would be determined entirely by particles with $p \cong p_0$, and its shape would be approximately that for single-particle radiation; thus neither case would satisfactorily explain a typical observed power-law radiation spectrum as the result of a steady-state combination of stochastic acceleration and radiation loss.

Third, the form of diffusion coefficient which can account for a power-law spectrum of particles (and so of radiation also) is clearly $D_{\text{eff}} \sim p^3$, and the resulting spectrum exponent is

$$n = \frac{2Kp^3}{3D_{\text{eff}}} - 2, \quad (5.3.5)$$

but this n must be greater than two. As was pointed out in Chapter 3, the relativistic variation of gyrofrequency with energy may make such a D_{eff} possible, but there would still be two weaknesses in such a model. First, the larger the range in p over which F is to vary as p^{-n} , the more critical it is that the exponent of p be very close to three. Second, this is subject to the same objection as the original Fermi model, that the exponent n is determined by a ratio of two apparently unrelated quantities; then the observation of values of n quite similar to one another and all of the order of unity must be interpreted either as accidental or as the result of a deeper relationship which remains to be elucidated.

Finally, for $D_{\text{eff}} \sim p^m$ with any $m > 3$ the formula (5.3.4) breaks down, giving an essential singularity for $p \rightarrow 0$ and an unnormalizable flat spectrum for $p \rightarrow \infty$. This means that in order to consider such a case one would have to explicitly take into account the eventual dropoff of D_{eff} for large p .

IV. Loss of Particles

An effect which has not been taken into account in any of the discussion above is the loss of energetic particles from the system in question. This might represent inelastic collisions (nuclear or elementary-particle interactions) in which one of the particles represented by the distribution function F suddenly loses an appreciable fraction of its energy or even disappears (with particles of some other kind appearing instead); or it might be the result of particles escaping from a finite region of space in which they were generated. These pictures were involved, separately, in Fermi's two principal papers on cosmic-ray theory;^{38,39} but a more general treatment than his will be given here. We shall continue in this section to suppose that there is sufficient scattering to keep the distribution function isotropic, so that only its dependence on total momentum p (or energy) must be considered.

The first case we shall take up is that of the loss of particles which are being stochastically accelerated. If $\alpha(p)$ represents the probability per unit time that a particle of momentum p will be lost, the equation representing these two processes is

$$\frac{\partial F}{\partial t} = \frac{1}{p} \frac{\partial}{\partial p} \left[p^2 D_{\text{eff}} \frac{\partial F}{\partial p} \right] - \alpha F . \quad (5.4.1)$$

The general steady-state solution of this equation cannot readily be written, and only a very limited investigation will be made here. Suppose that all the functions in this equation are approximated by power laws over some range of momenta:

$$D_{\text{eff}} = bp^m , \quad \alpha = ap^{m'} , \quad F \sim p^{-(n+2)} . \quad (5.4.2)$$

A steady state requires that the exponents have the values

$$m' = m - 2 , \quad m = n + 1 - \frac{a/b}{n + 2} , \quad (5.4.3)$$

so that the relative importance of D_{eff} with respect to α must increase strongly with p to support the particle distribution in this state. For given m , $n \geq m - 1$ and increases with a/b , which demonstrates the expected steepening of the distribution function by the loss term. Further comments must depend on consideration of what values of m and m' are within reason.

If a radiation term [as in (5.3.1)] is added to (5.4.1), the above procedure is limited to the case $m = 3$, for which

$$n = \frac{K}{3b} + \sqrt{\left(\frac{K}{3b} - 2\right)^2 + \frac{a}{b}} \geq 2 . \quad (5.4.4)$$

In the limit $a \rightarrow 0$ this agrees with, and slightly clarifies, equation (5.3.5).

For the contrasting case of loss of "injected" particles, the function $\alpha(p)$ may be generalized to $\alpha(p, \tau)$, the probability per unit time that a particle of momentum p and age τ since injection will be lost. If only the injection and loss terms are taken into account the problem is trivial, so radiation will also be included now. In either of two limits this case may be solved explicitly. First, consider $\alpha(p, \tau) \rightarrow \alpha(\tau)$, where the probability of loss depends only upon age; this description may apply when the loss represents escape from a finite region of space and may be approximately characterized by a transit time from point of injection to boundary. Then the steady-state solution is

$$F(p) = \frac{3}{2Kp^4} \int_p^\infty dp' p'^2 Q(p') \left[1 - \int_0^{\tau(p', p)} d\tau' \alpha(\tau') \right]; \quad (5.4.5)$$

its derivation is omitted because it is a straightforward generalization of (4.4.3). The new quantity in brackets is simply the probability of survival of a particle for the time

$$\tau(p', p) = \frac{3}{2} \left[\frac{1}{Kp} - \frac{1}{Kp'} \right] \quad (5.4.6)$$

in which the injection momentum p' would decrease to p under the influence of the radiation loss. In particular, it might be supposed that the escape process is approximated by assigning the same lifetime τ_0 to every particle; this would make $\alpha(\tau) = \delta(\tau - \tau_0)$ and

$$F(p) = \frac{3}{2Kp^4} \int_p^{p_0} dp' p'^2 Q(p') \quad (5.4.7)$$

with

$$p_0 = \frac{p}{1 - 2Kp\tau_0/3}. \quad (5.4.8)$$

Such a model was used by Ekspong, Yamdagni, and Bonnevier in the latter part of their article¹⁰² to investigate the effect of particle loss upon the spectra they had previously calculated on the basis of (4.4.3), as noted at the end of Chapter 4.

In the opposite limit where $\alpha(p, \tau) \rightarrow \alpha(p)$, the equation

$$\frac{\partial F}{\partial t} = \frac{2K}{3p} \frac{\partial}{\partial p} [p^4 F] + Q - \alpha F \quad (5.4.9)$$

has the steady-state solution

$$F(p) = \frac{3e^{g(p)}}{2Kp^4} \int_p^\infty dp' p'^2 e^{-g(p')} Q(p') \quad (5.4.10)$$

where

$$g(p) = \frac{3}{2K} \int_p^\infty dp' p'^{-2} \alpha(p') . \quad (5.4.11)$$

The meaning of this result may be clarified by writing instead the local effective value of the energy spectrum exponent,

$$n_{\text{eff}}(p) = -\frac{p}{F} \frac{dF}{dp} - 2 = 2 - \frac{3\alpha(p)}{2Kp} + \frac{p^3 Q(p)}{e^{g(p)} \int_p^\infty dp' p'^2 e^{-g(p')} Q(p')} . \quad (5.4.12)$$

The middle term shows that $\alpha(p)$ tends to decrease n_{eff} , and has the simple interpretation that each particle removed is one which would otherwise contribute to the spectrum at immediately lower energies, so that the spectrum must be flattened by this loss at the value of p where it occurs. But the last term shows the opposite effect of increasing n_{eff} along with $\alpha(p)$, because if the flux of particles from higher energies

is cut down then the local injection $Q(p)$, which is a steepening influence, will become relatively more important. Thus no general statement can be made about how loss of particles according to $\alpha(p)$ will influence the shape of the spectrum. Study of (5.4.5) shows that the same is true of loss according to $\alpha(\tau)$.

The only circumstance in which (5.4.12) will yield an actual power-law spectrum (n_{eff} independent of p) is when $\alpha(p) = ap$, as was the case with (5.4.4). But this dependence is exactly such that its only effect is a decrease in the magnitude of F by a factor

$$\frac{n_o - 1}{n_o - 1 + (3a/2K)} ,$$

with the exponent retaining the value $n_{\text{eff}} = n_o + 1$ which it would have in the absence of loss, as shown by (4.4.4).

V. Transients, Total Energies, and Ratios

This section contains brief comments on a few miscellaneous topics. First, there is the question of time-dependent spectra, since the preceding sections have emphasized the description of steady states. The qualitative answer follows from inspection of the various terms used in equations (4.4.1), (5.3.1), and (5.4.1). An increase in the rate of injection $Q(p)$ or a decrease in the loss rate $\alpha(p)$ will clearly allow $F(p)$ to increase in time toward some higher equilibrium value, but the effect of the other two terms is slightly more complicated. In the normal case for which F is decreasing and concave upward as a function of p , an increase in $D_{\text{eff}}(p)$ will tend to increase F in time, but an increase in $\partial D / \partial p$ will make F decrease in time at a given p ; this must not be confused with the effect of D on the total energy of all particles, as shown below. The effect of a departure of $\partial F / \partial p$ from what would be an equilibrium value is not uniquely connected to $\partial F / \partial t$, for it enters in opposite ways into the radiation and acceleration terms. A "young" increasing spectrum could be overly steep where acceleration dominates, the steepness itself enhancing the relative importance of the

acceleration; or it could be overly flat under the predominant influence of radiation, the increase in time at a given p being due to the downward flux in energy of an excess of higher-energy particles. Some numerical calculations of transient spectra are included in Kellermann's article;¹⁰¹ they demonstrate various effects of radiation upon particles which are injected in repeated bursts, all of which have been dealt with at least qualitatively in this and the preceding chapter.

Another quantity which may sometimes be of interest is the total kinetic energy density contained in all particles with energies above some given value. This is defined by

$$\mathcal{E}(> p_o) = \int d\Omega \int_{p_o}^{\infty} dp p^2 (\gamma-1)mc^2 F(\vec{p}) , \quad (5.5.1)$$

and if attention is limited to the relativistic region $p_o \gg mc$ and to an isotropic distribution function, it is easily found that

$$\begin{aligned} \frac{1}{4\pi c} \frac{\partial \mathcal{E}(> p_o)}{\partial t} = & p_o^3 \left[\left(\mathcal{F}(p_o) - \frac{2}{3} K p_o^2 \right) F(p_o) - D_{\text{eff}}(p_o) \frac{\partial F}{\partial p}(p_o) \right] \\ & + \int_{p_o}^{\infty} dp p^2 \left[pQ(p) + \left(\mathcal{F}(p) - \frac{2}{3} K p^2 - p\alpha(p) \right) F(p) - D_{\text{eff}}(p) \frac{\partial F}{\partial p}(p) \right] . \end{aligned} \quad (5.5.2)$$

As an alternative to injection or stochastic acceleration, the term $\mathcal{F}(p)$ has been added here to represent any steady, nonrandom force which may be working to accelerate particles; all other symbols have appeared in previous equations. If it were desired to consider \mathcal{F} in any of the preceding sections, this could be done by noting that it always enters the equations in the same way as does $-(2/3)Kp^2$.

Finally, we shall have occasion in the next chapter to discuss whether some information may be found about the relative magnitude, shape, and/or total energy content of the distribution functions for two different species of particles in the same region of space, when neither function

is known individually in detail. For the purpose of eventual further exploration along that line, we derive an equation here for the ratio of proton and electron distribution functions,

$$R(p, t) = \frac{F_p(p, t)}{F_e(p, t)} ; \quad (5.5.3)$$

both functions are being assumed isotropic again for simplicity, in the "model term" sense mentioned at the beginning of this chapter. "Injection" terms are not well suited to the present argument and are omitted, and only energies well above one BeV are considered, so that the two species will have essentially identical kinematic properties. Then the accelerating force $\mathcal{F}(p)$ will be the same for both; if waves of opposite circular polarizations and z-velocities have equal strengths, $D_{\text{eff}}(p)$ will also be the same; the loss rate $\alpha(p)$ would be the same for loss by escape, but insofar as it represents loss by nuclear reaction it may be quite different; and since K depends upon rest mass as the negative fourth power, the radiation term may be set to zero for the protons. Then the distribution functions in these approximations satisfy

$$\frac{\partial F_j}{\partial t} = \frac{1}{p} \frac{\partial}{\partial p} \left[p^2 \left(\frac{2}{3} \delta_{je} K p^2 - \mathcal{F}(p) \right) F_j \right] + \frac{1}{2} \frac{\partial}{\partial p} \left[p^2 D_{\text{eff}}(p) \frac{\partial F_j}{\partial p} \right] - \alpha_j(p) F_j . \quad (5.5.4)$$

Now the equation for electrons ($j = e$) is multiplied by $R(p, t)$, and in the equation for protons ($j = p$) F_p is replaced by $R F_e$; the difference between these two results gives the desired equation for R , with only a weak dependence on F_e :

$$\frac{\partial R}{\partial t} = - \frac{2}{3} K p \left(4 + p \frac{\partial \ln F_e}{\partial p} \right) R - \left(\mathcal{F} - 2 D_{\text{eff}} \frac{\partial \ln F_e}{\partial p} \right) \frac{\partial R}{\partial p} + \frac{1}{2} \frac{\partial}{\partial p} \left[p^2 D_{\text{eff}} \frac{\partial R}{\partial p} \right] - (\alpha_p - \alpha_e) R . \quad (5.5.5)$$

A limitation on the use of this equation is that a reasonable boundary condition must be supplied at some minimum energy. A similar procedure might be followed for the ratios of numbers of heavier particles of charge Z to number of protons (for application to the cosmic-ray abundance problem), but it would be made more difficult by the dependence of \mathcal{F} and (especially) D_{eff} upon Z .

Chapter 6

ASTROPHYSICAL APPLICATIONS

I. Introduction

The theory developed in the preceding chapters is now to be compared with the observational evidence about the actual occurrence of high-energy particles in nature. The main emphasis of this discussion will be upon quasars, because of their great current interest, because of the important unsolved problems they present, and because they seem to be the most outstanding examples of the general high-energy phenomenon. Radio galaxies, supernovae, cosmic rays, etc., will be mentioned in parallel whenever it seems that they can aid our understanding.

When this study was undertaken, it was not anticipated that such a full development of the abstract stochastic acceleration problem would be included. But since that has proved necessary, there has been a corresponding reduction in the effort which could be devoted to the material of this chapter, so it will be understood that this is only a preliminary treatment of the applications. That is, part of the original program remains to be completed, and one of the most important goals of this chapter is to indicate in what directions further study should proceed in order to realize the full potential of both the theory and the observational data.

II. Spatial Structures

The function of this section will be largely to provide some foundation for the following one, where there will be a more direct relevance of the data to the models of the preceding chapters. Only the origin of the continuous spectrum of electromagnetic radiation is considered here; the emission and absorption lines present a formidable problem by themselves, and they seem to arise mostly in regions and conditions which differ from those for the continuum. There also seems to be an anticorrelation between the presence of absorption lines and of the variations which are to be considered below.¹⁰³ Thus the following remarks will concern the central "point sources," the optical jets or filaments associated with a few of these objects, and the external radio clouds.

By "point source" we mean the very compact central object referred to by the optical identification of a quasar, which apparently is the site of the original explosion that produces the other phenomena. We shall not attempt to use any of our results above to account for the high-frequency radiation of the point source, for it probably cannot be synchrotron radiation. This has been argued on the basis of competition from the inverse Compton process by Hoyle, Burbidge, and Sargent.^{104,48} Also, Hazard, Gulkis, and Bray¹⁰⁵ have pointed out that Dent's observation of variation at 8000 Megacycles in the source 3C273B,¹⁰⁶ together with the absence of self-absorption down to at least 410 Mc, implies (on the assumption of cosmological distance) either that there is a remarkably great change in angular size of the source between frequencies 1420 and 8000 Mc or else that the radiation cannot be from the synchrotron mechanism. Ginzburg and Ozernoi⁵³ have presented a case for a collective radiation mechanism, which deserves further investigation; this seems to be the only reasonable way, if the quasar redshifts are cosmological, to account for the observed high brightness temperatures.¹⁰⁷

The jet associated with 3C273 is the most striking example of a phenomenon also observed in such sources as 3C48, 3C279, 3C287, and M87. These provide important evidence about the ages of these objects and about the violent nature of the underlying explosions. They can be interpreted somewhat independently of the more drastic conditions in the nucleus of a quasar, and their radiation definitely seems due to the synchrotron mechanism. Recent observations of polarizations at 6-cm wavelength indicate¹⁰⁸ an average magnetic field parallel to the jet or nebulosity in each of the sources mentioned above. Similar evidence has been obtained at optical frequencies by Hiltner¹⁰⁹ and by Kinman.¹¹⁰

Besides the radio emission associated with the "point source," there are sometimes one or more associated "radio clouds." These show the existence outside the nuclei of limited regions which contain turbulent plasma; they present the problem of deciding whether they result entirely from the directional nature of the original explosion, or whether they are also subject to confinement, either in the usual sense of plasma containment by magnetic fields or perhaps by the hydromagnetic self-attraction suggested by Parker for interstellar clouds.¹¹¹ Our discussion will

not go into the question of the effect upon the radio spectrum of adiabatic deceleration in such a cloud if it should be expanding, for this has been dealt with by Kardashev⁶⁷ and by van der Laan.¹¹²

Many of these phenomena are paralleled, as we emphasized in Chapter 1, in other objects. We may note in addition here that, besides an extended turbulent, filamentary structure radiating an intense synchrotron continuum, the Crab Nebula also has a "point source" which is optically obscured but has been the subject of recent investigations at radio frequencies. This region also seems to require interpretation with a collective emission mechanism.⁵³

III. Spectral Properties

The most immediately outstanding thing about the spectra of extragalactic radio sources is their nonthermal nature. Most of these spectra may be fitted quite well by a power law in frequency, with different values of the spectral index occurring about as shown in Fig. 8. This

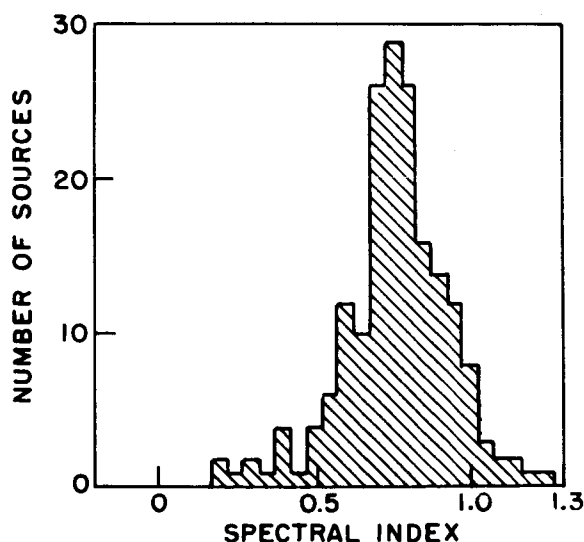


FIG. 8. SPECTRAL INDICES OF EXTRAGALACTIC RADIO SOURCES, FROM KELLERMANN.

histogram is taken from Kellermann¹⁰¹ and is nearly the same as his earlier version;¹¹³ it represents all high-galactic-latitude sources, whether optically identified or not, with the spectral indices being fitted over the range 38-1400 Mc. Figure 9 shows a similar distribution for the more limited sample obtained by using only 66 sources known to be quasars, and

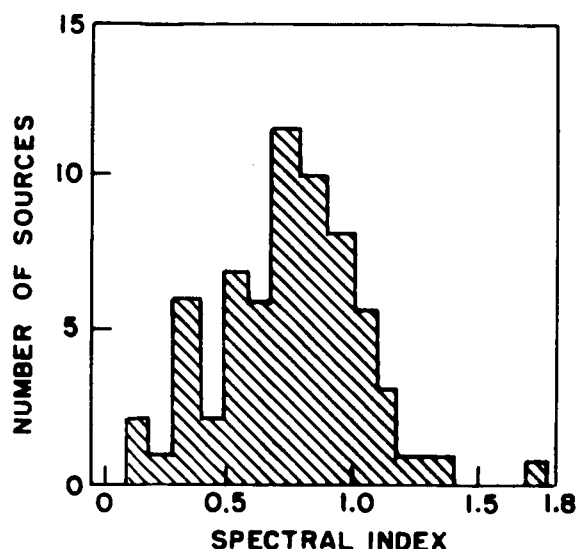


FIG. 9. SPECTRAL INDICES OF QUASARS.

limiting the fitting of the spectral indices to higher frequencies to avoid inclusion of points lowered by self-absorption. Indices fitted between 750 and 1400 Mc were available for 46 of these from the NRAO catalog,¹¹⁴ and for the remaining 20 an index for either 408 or 960 to 1410 Mc was calculated from flux data compiled by P. Feldman from several sources. The spectral indices of quasars are seen to be quite similar to those of extragalactic sources as a whole. But the significance of these histograms should not be overestimated, for they do not adequately represent the considerable fraction of sources which show curvature (on a $\log I$ vs $\log \nu$ plot) or variability in their spectra. The radio spectra of a few of these sources are known up to 8000 Mc;¹¹⁵ 3C273 is still the exception in having fairly detailed observations of its optical continuum¹¹⁶ and some isolated information in the infrared¹¹⁷ and millimeter^{118,119} ranges.

The second important spectral property is variability; this has now been observed with certainty in at least five quasars and with high likelihood in eight more,¹⁰³ although there are others for which repeated observations have not detected any change. This property is also shared by at least one radio galaxy, the Seyfert galaxy NGC 1275.¹²⁰ The two most notable examples are 3C273, for which optical fluctuations over at least the last 75 years have been established from the Harvard plate collection,¹²¹ and 3C446, which in the late summer of 1966 exhibited irregular

fluctuations over a range of two or three magnitudes at rates as high as 0.2 magnitudes per day.¹²² In the latter case, the optical polarization has also been observed to vary, apparently in close correlation with the changes in total magnitude.¹¹⁰ Detailed information on the radio variations of several objects has recently been given by Pauliny-Toth and Kellermann.¹²³

The model with which Kellermann¹⁰¹ attempts to explain these spectral indices and variations calls for the existence of some process capable of injecting isotropic bursts of energetic electrons with a power-law spectrum equivalent to $n = 1.5$:

$$Q(p, t) = A p^{-3.5} \delta(t - t_i), \quad i = 0, 1, 2, \dots \quad (6.3.1)$$

Each of the series of injection times t_i is supposed to follow the preceding one by an interval of the order of some definite scale T_0 . Then sources with spectral index 0.25 are explained as exhibiting radiation from a fresh burst according to equation (4.1.2), being observed within a shorter time after the burst than the radiative lifetime at the observation frequency. The preponderance of sources with spectral index 0.75 is explained as showing equilibrium between injection and radiative loss, according to equation (4.4.5); the radiative lifetime is supposed to be greater than T_0 so that the bursts of particles may be approximated by a continuous injection. Finally, sources with spectral index 1.33 are explained according to equation (4.3.15); it is supposed that, at the time and frequency of observation, the elapsed time since the last burst has been greater than the radiative lifetime of a particle with pitch angle $\pi/2$ but still less than T_0 . This model suggests that the radiation of sources with flat spectra may be likely to vary irregularly; for those with "normal" spectra, it may be fairly constant in time; and for those with very steep spectra, it should be smoothly decreasing, following equation (4.3.16).

The positive accomplishments of this model are (1) to show some causal relationship among the three apparently significant numbers 0.25, 0.75, and 1.3; (2) to predict the association of flat spectra with variability, which appears to be true; and (3) to suggest a development in time for the spectral

contribution of a new burst which fits qualitatively with the available data.¹²³ Besides the fact that it makes no pretension of accounting for the origin of the spectrum of the injected particles, this explanation has some other possible weaknesses: (1) the injection process is not likely to be isotropic; (2) the uniformity of field strength is a relatively innocuous assumption, but it would still be desirable not to depend on it; (3) there is a considerable spread of indices around the canonical values, so the exponent 1.5 for the energy spectrum of injected particles would be universal in only an approximate sense; (4) the 0.25 explanation of flatter spectra still seems somewhat unnatural to this writer, and Kellermann actually uses self-absorption instead in explaining the shapes of transient components; (5) the inadequacy of present observations is no fault of the model, but still prevents a conclusive comparison of detailed predictions; and (6) the 1.33 explanation of steeper spectra seems incompatible with our belief that the plasma is turbulent. In order to show this last point, the time scale for magnetic scattering may be compared with that for radiative energy loss, just as it was compared with the Coulomb scattering time in equation (3.5.5). For illustration, consider a 5×10^{10} eV electron radiating at centimeter wavelengths in a field of order 10^{-4} gauss, with fluctuation amplitudes of 10 percent; then for $n = 2$ in equation (3.5.3) it is found that the scattering time will be less than the radiation time for any value of the fluctuation coherence time T_C between about 10^{-7} and 10^{11} seconds.

Therefore, without detracting from the value of Kellermann's suggestions, we believe that at this stage, the construction of alternative models should still be pursued. Any possible explanation for the "injected" $W^{-1.5}$ spectrum must also be carefully studied; hence we remark here on the cases in previous chapters for which power-law spectra were found. First, equation (5.3.5) may be dismissed because it can only account for an exponent greater than two, and because the description of this process above as an injection is incompatible with its being the result of an equilibrium involving radiation. It might be more reasonable to use equation (5.4.3), for an instability could be driving a strong acceleration in a small and leaky region, the leakage constituting the

injection into a much larger surrounding region. To obtain $n = 1.5$, the exponent of the stochastic acceleration coefficient in (5.4.2) should be

$$m = 2.5 - \frac{a/b}{3.5} = 2.5 - \frac{p^2 \alpha / D_{\text{eff}}}{3.5} . \quad (6.3.2)$$

But here again there is the problem of explaining why the ratio a/b should always be the same. One possible answer is that the dependence is only weak and the canonical 1.5 only approximate; for example if $m = 2$ then values of a/b ranging from zero to four would only cause n to vary between one and two, and the corresponding steady-state radiation spectrum indices of 0.5 to 1.0 might easily fit the distribution of Fig. 9. Alternatively, one might argue that the acceleration alone determines the spectrum; that is, $a/b \ll 1$, and $m = 2.5$ would be interpreted as a unique number which should be predictable when the natural spectrum of plasma turbulence is sufficiently well understood.

Finally, it may be argued that this injection should not be thought of merely as a steady state of processes operating on a shorter time scale, but as probably involving time-dependent spectra. If there is still to be something unique about the spectral shape, self-similar spectra are the ones to be considered. If stochastic acceleration alone is considered, again with

$$D_{\text{eff}} = b p^m , \quad (6.3.3)$$

then there are self-similar solutions

$$F(p, t) \propto t^{-\frac{3}{2-m}} \exp \left[- \frac{p^{2-m}}{(2-m)^2 b t} \right] ; \quad (6.3.4)$$

these are generalizations of (3.4.1). They are of the wrong shape to be of interest, except in the singular case $m = 2$, for which there appears an additional undetermined constant C :

$$F(p, t) \propto e^{Ct} p^{-(n+2)} \quad (6.3.5)$$

with

$$n = \frac{1}{2} \left[-1 + \sqrt{9 + 4C/b} \right]. \quad (6.3.6)$$

The only way to determine a value of n here is to refer to another level of "initial conditions," so this is not a satisfactory explanation.

Our preliminary conclusion on this matter is that two, and perhaps three, processes are important in determining these nonthermal radio spectra: First, stochastic acceleration is reasonable both as to estimates of acceleration rate¹²⁴ and as to its dependence on particle energy (from arguments immediately above and in Section VI of Chapter 3), while the existence of any alternative coherent acceleration process which could account for these properties seems practically impossible. Second, we believe that the radiation loss plays an important role in this determination, and must be included in any model which might extend or replace Kellermann's. Third, loss of particles by escape or nuclear reaction is probably not a prime determinant, but as was noted above, it could well account for a spread of spectral properties about the canonical values that would otherwise result from the acceleration and radiation loss.

IV. Maximum and Total Energies and Cosmic Rays

The maximum energy which any particles may be expected to attain will depend in general not only on the nature and strength of the accelerating and decelerating processes, but also on the length of time they have been in operation. This is illustrated on page 9 of Reference 124, where the stochastic acceleration of electrons in the jet of 3C273 is estimated by a simple random-walk argument. A total of 10^9 steps over a period of 10^5 years leads to an "average energy gain" of 10^{13} eV. But a 10^{13} eV electron in the assumed field of 4×10^{-5} gauss will lose energy by radiating at about 10^2 eV sec⁻¹, so the "average particle" will never attain much over 10^{12} eV; but this is still sufficient, even rather liberal, to account for the observed optical synchrotron radiation.

The same arguments should apply to a proton, without any reduction for radiation loss. But we wish to point out a pitfall in another argument that was given by Sturrock to estimate acceleration rates.¹²⁵ There it was said that $10^{12.3}$ eV electrons continue to be present, although radiating at $10^{1.3}$ eV sec⁻¹, so that there must be an acceleration process working upon them at the same rate; then this acceleration rate is applied to protons also, to estimate that they would achieve energies in excess of $10^{13.8}$ eV. But this is essentially a single-particle estimate, and it should be kept in mind that if this acceleration is of a statistical nature, its effective rate of action will depend upon what spectrum of particles has already been generated. By referring to equation (5.5.2), it may be seen that the rate of energy gain from stochastic acceleration is proportional to $-(\partial F / \partial p)$, so that an electron spectrum which was steepened by radiation loss could be absorbing more power per particle on the average than the proton spectrum which would develop under the same fluctuating fields. On the other hand, one must not be misled by casual inspection of (5.3.1) into thinking that this average acceleration rate is ever likely to vanish; for even if the protons should attain a stationary spectrum ($\partial F / \partial t = 0$) over some range of momenta, there would still be a nonzero current of particles flowing outward in momentum space.

Since it appears that protons with energy of 10^{13} eV or more may be generated in the jet of 3C273, one may surmise that the nuclei of quasars could have sufficiently stronger turbulent fields to accelerate protons to energies one or a few orders of magnitude greater than this; but it is more difficult to make direct estimates. The possibility remains open that these objects could be significant generators of primary cosmic rays. This calls for closer study of several points, as we shall outline in the remainder of this section.

First, there must obviously be greater certainty about the maximum attainable energies which were estimated above. In the present state of the field, no estimate which depends on detailed analysis of a particular model of quasars is likely to be widely accepted. But if one attempts to make an estimate which will depend entirely upon observed quantities, not only is there a scarcity of the kind of data that would be needed, but as long as the local-cosmological dispute remains unsettled there will not even be agreement upon what has really been observed.

Second, the proper particle spectrum must be accounted for, and this may still be the most difficult part of the problem. The Kellermann model would suggest injection of protons with the same spectrum as the injected electrons,

$$N(W) \propto W^{-1.5}, \quad (6.4.1)$$

but the observed cosmic ray spectrum varies as $W^{-2.5}$ (see Chapter 1, Section IID). The electron spectrum is eventually steepened to an exponent 2.5 by radiation losses, but the same process for protons would require very strong magnetic fields and does not seem to be a reasonable possibility.¹²⁴ The probability for a particle to escape from the generating object into intergalactic space would depend on energy, but this would be expected only to flatten rather than steepen the spectrum. Thus it appears that cosmic ray protons and radiating electrons will not be naturally explained as both being products of the same set of processes and conditions, unless the Kellermann model is somehow altered or replaced.

Third, the question of the relative total energies gained by all electrons and by all protons must be studied. Let the ratio be⁹⁷

$$\kappa_r = \frac{\mathcal{E}_{\text{protons}}}{\mathcal{E}_{\text{electrons}}}. \quad (6.4.2)$$

The value $\kappa_r \approx 100$ has often been used, and is based on the relative abundance of electrons in the cosmic rays observed in the vicinity of the earth; but this involves accounting for the electrons as "secondaries," or products of nuclear collisions of the positively charged primary nuclei, and does not seem a relevant estimate when the simultaneous generation of both species as "primaries" is being considered. There are, however, other reasons which we may describe qualitatively for expecting κ_r to be much greater than unity: Other things being equal, electrons are subject to radiation loss and protons are not, which ought to leave the protons with greater total energy; and the "selection rules" under which the acceleration begins seem to strongly favor the protons. Referring to the

discussion in Sections IV and VI of Chapter 3, we see that the electrons as they start from nonrelativistic energies have a much higher gyrofrequency than the protons and so are accelerated by waves of higher frequency and presumably smaller amplitude. Furthermore, they may have difficulty as their gyrofrequencies decrease in crossing the gap where the fluctuation spectrum is depleted by acting upon thermal protons; and only above 10 BeV may they be expected to gain equal footing with the protons. Yet other reasons may be cited^{45,107} for believing that $\kappa_r \lesssim 1$, so a more careful weighing of these arguments will be necessary.

Finally, the total rate of production by the whole class of sources must be computed; this may be done for quasars in the same way it was done earlier for radio galaxies by Burbidge and Hoyle.⁶⁰ There have now been around 100 quasars observed; suppose that these represent a complete sampling out to a distance corresponding to redshift $z = 2$. For a Hubble constant $H = 100 \text{ km sec}^{-1} \text{ Mpc}^{-1}$ this would be about $10^{28.3} \text{ cm}$, and would make an average of $10^{-85.5}$ quasar per cubic centimeter. Taking figures suggested by Pauliny-Toth and Kellermann,¹²³ suppose that each active quasar is repeatedly producing $10^{58.5}$ ergs in bursts of relativistic electrons repeated on a scale of 10 years, or $10^{8.5}$ seconds. As we remarked above, the ratio of proton and electron energies produced in these bursts is uncertain, but let us use the value $\kappa_r = 1$, which is hopefully a conservative estimate. Then the average rate of production of high-energy protons would be $10^{-35.5} \text{ erg sec}^{-1} \text{ cm}^{-3}$, and if this were to accumulate (and be evenly distributed) over the "lifetime of the universe," about 10^{10} years, it could account for an energy density of around $10^{-18} \text{ erg cm}^{-3}$. This is considerably below the $10^{-12} \text{ erg cm}^{-3}$ observed for cosmic rays near the earth, and also compares unfavorably with the Burbidge and Hoyle estimate of $10^{-13 \pm 1} \text{ erg cm}^{-3}$ for radio galaxies.

Although redshifts have only been determined for about 100 quasars, the idea is again being advanced¹²⁶ that there are some 10^5 of these objects over the whole sky (down to blue magnitude 19.7), but that most of these have been observationally discriminated against because they are radio-quiet. If one were prepared to believe that all of these objects generate energy at the same rate as those with "active" radio spectra, the estimate above for the cosmic-ray energy density due to quasars could

be increased to 10^{-15} erg cm⁻³. This is still quite low, but could possibly be a significant contribution to a "universal cosmic ray distribution" which, according to some models, is being observed at energies above the cutoff for the Galactic cosmic rays at 10^{15} or 10^{16} eV.

V. General Conclusions and Suggestions

1. We believe that stochastic acceleration in some form is important in the generation of high-energy particles in nature, and that this will form an integral part of the ultimately accepted explanation; in particular, it seems likely that this stochastic acceleration is to be connected with plasma turbulence. Not only is there a lack of reasonable alternatives, but we believe that the present work has made some steps toward showing that the stochastic mechanism is itself quite reasonable. We think it is worthwhile to begin forging additional links necessary to this chain of reasoning: first, to classify modes of oscillation in a magnetized plasma and to specify for each class the relation of field amplitudes to mode amplitudes for arbitrary polarizations and directions of propagation; and second, to search for a theory of mode amplitudes in a quasi-equilibrium state. The latter problem will require a thorough understanding of both similarities and differences between the turbulent states of plasmas and of ordinary fluids,¹²⁷ and may well profit from a partly numerical approach.¹²⁸

2. Although we have stressed the problem of quasars in this work, we are hopeful that our ideas may be of use in several sites of application, such as those outlined in Chapter 1. Evidence for the occurrence of explosions in galactic nuclei has been reviewed by Burbidge, Burbidge, and Sandage;¹²⁹ and we are inclined toward positive views, somewhat similar to those of Shklovskii,¹³⁰ in regard to the possibility of a close generic relationship among quasars and radio, Seyfert, and normal galaxies.

3. For reasons given in Chapter 3, we have suggested that stochastic acceleration is most likely to occur as a transverse cyclotron-resonance effect of low-frequency (hydromagnetic) waves, rather than being connected with electromagnetic or longitudinal ("plasma wave") modes.

4. Since only one phase of the problem was selected for treatment in this work, the origin of the "magic exponent" 1.5 or 2.5 for energetic-particle spectra has not been fully explained. But one should now be prepared to watch carefully as theories of plasma turbulence are developed for a natural spectrum of fluctuations

$$S \sim \omega^{-n},$$

with either $n = 2.5$ precisely, or else $n \lesssim 2.5$. An explanation of the observed radiation spectra in terms of such a turbulence spectrum would immediately be suggested in either case by the discussion following equation (6.3.2).

5. Although the conclusions in Section IV of this chapter do not encourage the belief that acceleration of protons in quasars accounts for the major portion of the observed cosmic rays, our line of reasoning suggests that the acceleration in quasars is a manifestation of a universal spectrum of turbulence which is also present in the objects (e.g., supernovae) which do supply the cosmic rays. Therefore it is still of great interest to find some way of fitting the acceleration of electrons to a $W^{-1.5}$ spectrum and protons to $W^{-2.5}$ both into the same picture.

6. In attempting to remove the test-particle restrictions of the present model, one should consider the possibility of simply estimating a rate of depletion of plasma turbulence energy and using this heuristically to delineate the circumstances for which this depletion is an important determining factor. This would be somewhat similar to Parker's use of the concept of a "cosmic ray gas" coextensive with the ordinary interstellar gas, with which he has shown that cosmic rays have an effective pressure which is important in certain galactic processes.¹³¹

7. As data accumulate in the new fields of X- and gamma-ray astronomy, some of it will bear upon the problems we have considered here, and should be studied accordingly.

8. As to what further observations would be helpful, we can only urge several things that are already obvious: (a) coordinated observations of variable radio sources at more frequent time intervals and more closely spaced spectral frequencies, in order to obtain something more

nearly approaching a "sweep-frequency record" of their outbursts; (b) better establishment of radiation spectra at microwave and infrared frequencies, and particularly in the difficult region between these; and (c) laboratory study of steady weak turbulence in plasmas, especially that directed toward establishing the form of the natural turbulence spectrum.

9. Although we have felt it possible to formulate some correct ideas within this limited framework, one's mind should be kept open, as more detailed theories become possible, toward the inclusion at appropriate points of several effects which we have omitted. These might include bremsstrahlung and collective effects⁵³ (particularly for the nuclei or "point objects" and for high frequencies), energy loss in adiabatic expansion,⁶⁷ and (for low frequencies) inverse bremsstrahlung, synchrotron self-absorption and the "Tsytovich effect."¹⁰⁰ Finally, one additional effect of possible importance is the inverse Compton effect.⁴⁸ As a first approximation, this may be treated as synchrotron radiation in the magnetic fields of a flux of low-energy photons, and it is in this sense that some of our results above may still be used in the presence of this effect. But to be more exact, one should use further details of the correct theory presented by Felten and Morrison.¹³² Woltjer has attempted¹³³ to avoid the introduction of the inverse Compton effect by postulating electron injection only into a rather narrow cone about the direction of the magnetic field, but this idea would seem strained beyond reasonableness to account for the recent observations of Wampler on 3C446.¹³⁴

"That which is far off, and exceeding deep,

Who can find it out?"

--Ecclesiastes 7:24

Appendix A

COMPLEX VECTOR COMPONENTS

It proves convenient in some of our work to use the quantities

$$\hat{x}_+ = \hat{x} + i\hat{y}, \quad \hat{x}_- = \hat{x} - i\hat{y}. \quad (\text{A.1})$$

It should be kept clearly in mind that \hat{x}_+ and \hat{x}_- , as they have been chosen here, are not actually unit vectors, nor are they self-dual. But rather than carry along a distinction between covariant and contravariant vectors, we shall just follow the conventions listed below, in the spirit that this is a notational shorthand for which the only requirement is that everything turn out correctly whenever translated back to normal Cartesian components. If \vec{a} and \vec{b} are arbitrary vectors, we write

$$a_+ = a_x + ia_y, \quad a_- = a_x - ia_y, \quad (\text{A.2})$$

$$a_x = \frac{1}{2}(a_+ + a_-), \quad a_y = -\frac{1}{2}i(a_+ - a_-), \quad (\text{A.3})$$

so that

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z = \frac{1}{2}a_+ b_- + \frac{1}{2}a_- b_+ + a_z b_z. \quad (\text{A.4})$$

The thing potentially most confusing in this notation is the distinction between the μ -component of the gradient vector and the formal derivative with respect to the corresponding variable:

$$\frac{\partial}{\partial x_+} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) = \frac{1}{2} \nabla_-, \quad \frac{\partial}{\partial x_-} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) = \frac{1}{2} \nabla_+, \quad (\text{A.5})$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x_+} + \frac{\partial}{\partial x_-}, \quad \frac{\partial}{\partial y} = i \left(\frac{\partial}{\partial x_+} - \frac{\partial}{\partial x_-} \right), \quad (\text{A.6})$$

so that

$$\vec{a} \cdot \vec{\nabla} = a_+ \frac{\partial}{\partial x_+} + a_- \frac{\partial}{\partial x_-} + a_z \frac{\partial}{\partial z} . \quad (A.7)$$

Appendix B

SYMMETRY PROPERTIES

It is useful in several places to know a few special properties of the correlation and spectrum functions of stationary, homogeneous turbulence in a uniform plasma; these will be briefly summarized here. First, it follows immediately from the definition (3.1.5) that

$$R_{\alpha\beta}^{PQ}(\vec{\xi}, \tau) = R_{\beta\alpha}^{QP}(-\vec{\xi}, -\tau) , \quad (B.1)$$

and then from this and (3.1.7) that

$$S_{\alpha\beta}^{PQ}(\vec{k}, \omega) = S_{\beta\alpha}^{QP}(-\vec{k}, -\omega) . \quad (B.2)$$

Second, the fields \vec{E} and \vec{B} are real, so the correlation functions must be also except insofar as complex directions have been used in the subscripts:

$$\left[R_{\alpha\beta}^{PQ}(\vec{\xi}, \tau) \right]^* = R_{\alpha^*\beta^*}^{P^*Q^*}(\vec{\xi}, \tau) . \quad (B.3)$$

Then for the spectrum function we have

$$\left[S_{\alpha\beta}^{PQ}(\vec{k}, \omega) \right]^* = S_{\alpha^*\beta^*}^{P^*Q^*}(-\vec{k}, -\omega) = S_{\beta^*\alpha^*}^{Q^*P^*}(\vec{k}, \omega) . \quad (B.4)$$

In the case discussed in Chapter 3, Section III, where the statistical properties of the fields are assumed to have cylindrical symmetry, a rotation about the z axis amounts to a mere relabeling and must leave the spectrum function unchanged:

$$S_{\alpha\beta}(k_{\perp}, \varphi, k_{\parallel}, \omega) = S_{\alpha'\beta'}(k_{\perp}, \varphi + \psi, k_{\parallel}, \omega) . \quad (B.5)$$

Here α' and β' stand for those directions obtained by rotating vectors in the α and β directions through the angle ψ about the z axis;

for example, $\alpha = z \rightarrow \alpha' = z$ for any ψ , and $\alpha = x \rightarrow \alpha' = y$ for $\psi = \pi/2$. It is particularly convenient now to be working with the complex coordinates x_{\pm} , because the operation of rotation through the angle ψ is then accomplished merely by multiplying with a factor $e^{\mp i\psi}$ for each subscript \pm . By letting ψ take the value $-\varphi$ we can see that the entire angular dependence of $S_{\alpha\beta}$ is given in this case by

$$S_{\alpha\beta} \propto e^{i(a-b)\varphi}, \quad (\text{B.6})$$

where a is the number of $+$ and b the number of $-$ subscripts included in α and β . Then the angular part of $\int d^3k$ can immediately be carried out in (3.3.15), etc.

Appendix C

EVALUATION OF INTEGRALS

When (2.4.8) is being calculated in the presence of a magnetic field B_0 , we must evaluate integrals of the form

$$\begin{aligned}
 L(p, \theta) &= \frac{1}{2\pi} \int_0^{2\pi} d\phi \left\langle \Delta Q_\beta(\vec{x}'(t), t) e^{i\mu(\Omega t - \phi)} \int_{-\infty}^t dt' \Delta P_\alpha(\vec{x}'(t'), t') e^{i\nu(\Omega t' - \phi)} \right\rangle \\
 &= \frac{1}{2\pi} \int_0^{2\pi} d\phi \int_{-\infty}^t dt' R_{\alpha\beta}^{PQ}[\vec{x}'(t) - \vec{x}'(t'), t-t'] e^{i\mu(\Omega t - \phi) + i\nu(\Omega t' - \phi)},
 \end{aligned} \tag{C.1}$$

where μ and ν may be 0 or ± 1 and the present ϕ was called ϕ_0 in (3.3.4). In the limit of small gyroradius we could abbreviate all the arguments of R in this equation by $(t-t')$ and have a simple answer as with equation (3.2.4); but in the general case $\vec{x}'(t) - \vec{x}'(t')$, the separation between two points on a helix, given by (3.3.1) and (3.3.2), is not a function of the difference $t-t'$ alone. The use of the spectrum function gives a way of separating the field properties (in the correlation function R itself) from the particle-orbit properties (in the arguments of R):

$$\begin{aligned}
 L(p, \theta) &= \int d^3k \int d\omega S_{\alpha\beta}^{PQ}(\vec{k}, \omega) M(p, \theta, \vec{k}, \omega), \\
 M &= \frac{1}{2\pi} \int_0^{2\pi} d\phi \int_{-\infty}^t dt' \exp \left[i\mu(\Omega t - \phi) + i\nu(\Omega t' - \phi) \right. \\
 &\quad \left. + i\vec{k} \cdot (\vec{x}'(t) - \vec{x}'(t')) - i\omega(t-t') \right].
 \end{aligned} \tag{C.2}$$

We use cylindrical coordinates in k -space, with azimuthal angle ϕ , to write

$$\vec{k} \cdot (\vec{x}'(t) - \vec{x}'(t')) = 2k_{\perp} r_g \sin \left[\frac{\Omega(t-t')}{2} \right] \cos \left[\frac{\Omega(t+t')}{2} - \phi + \varphi \right] + k_{\parallel} v_{\parallel} (t-t') . \quad (C.3)$$

The variables are untangled by defining the new quantity

$$\phi' = \phi - \varphi - \frac{1}{2}\Omega(t+t') ; \quad (C.4)$$

all domains of width 2π in this variable are equivalent, and we have simply

$$M = \frac{1}{2\pi} \int_0^{2\pi} d\phi' \int_{-\infty}^t dt' \exp \left[i\mu \left(\frac{1}{2}\Omega(t-t') - \phi' - \varphi \right) + i\nu \left(-\frac{1}{2}\Omega(t-t') - \phi' - \varphi \right) + i2k_{\perp} r_g \cos \phi' \sin \frac{1}{2}\Omega(t-t') + i(k_{\parallel} v_{\parallel} - \omega)(t-t') \right] . \quad (C.5)$$

Now we have only to use the identity

$$e^{iu \sin \psi} = \sum_{m=-\infty}^{\infty} e^{im\psi} J_m(u) \quad (C.6)$$

to achieve complete separation of the two integrals. One of them,

$$\int_{-\infty}^t dt' e^{is(t-t')} = \pi \delta(s) - P(i/s) , \quad (C.7)$$

was already used in (3.2.5); $\delta(s)$ is the Dirac delta-function and P denotes "principal value" for subsequent integration over s . The other,

$$I_m^{\mu+\nu}(k_{\perp} r_g) = \frac{1}{2\pi} \int_0^{2\pi} d\phi e^{-i(\mu+\nu)\phi} J_m(2k_{\perp} r_g \cos \phi) , \quad (C.8)$$

is the $(\mu+\nu)^{\text{th}}$ Fourier component of a certain function over the domain $(0, 2\pi)$ and may be found as follows. Abbreviating $\mu+\nu$ by q and $k_{\perp} r_g$ by x , we use one form of "Bessel's Integral" to write

$$I_m^q(x) = \frac{1}{2\pi} \int_0^{2\pi} d\phi e^{-iq\phi} \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{i(m\theta - 2x \cos \phi \sin \theta)} . \quad (C.9)$$

Now we use the identity

$$2 \cos \phi \sin \theta = \sin(\phi+\theta) - \sin(\phi-\theta) \quad (C.10)$$

and (C.6) twice more to obtain

$$I_m^q(x) = (2\pi)^{-2} \int_0^{2\pi} d\phi \int_0^{2\pi} d\theta e^{i(m\theta - q\phi)} \\ \cdot \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} e^{-ir(\phi+\theta) + is(\phi-\theta)} J_r(x) J_s(x) . \quad (C.11)$$

The integrals here are merely Kronecker deltas requiring that

$$q + r - s = 0 , \quad m - r - s = 0 , \quad (C.12)$$

so q and m must be either both even or both odd for a non zero result, and it is surprisingly simple:

$$I_m^q(x) = J_{\frac{m-q}{2}}(x) J_{\frac{m+q}{2}}(x) . \quad (C.13)$$

Negative indices are always easily replaced by positive ones according to

$$I_m^{-q}(x) = I_m^q(x) = (-1)^m I_{-m}^q(x) . \quad (C.14)$$

There is an interesting curiosity which makes it quite easy to write the power series expansion of $I_m^q(x)$. The coefficients will be sums of products of Bessel function coefficients:

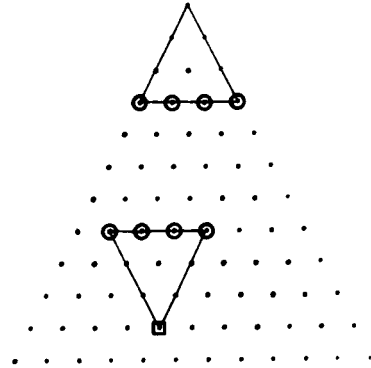
$$J_r(x) J_s(x) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{j+k} (x/2)^{r+s+2j+2k}}{j!(r+j)! k!(s+k)!} \\ = \sum_{u=0}^{\infty} \sum_{k=0}^u \frac{(-1)^u (x/2)^{r+s+2u}}{(u-k)!(r+u-k)! k!(s+k)!}$$

$$\begin{aligned}
&= \sum_{u=0}^{\infty} \left[\sum_{k=0}^u \binom{u}{k} \binom{r+s+u}{s+k} \right] \frac{(-1)^u (x/2)^{r+s+2u}}{u! (r+s+u)!} \\
&= \sum_{u=0}^{\infty} \binom{r+s+2u}{s+u} \frac{(-1)^u (x/2)^{r+s+2u}}{u! (r+s+u)!}, \quad (C.15)
\end{aligned}$$

so that

$$I_m^q(x) = \sum_{u=0}^{\infty} \binom{m+2u}{\frac{m+p}{2}+u} \frac{(-1)^u (x/2)^{m+2u}}{u! (m+u)!}. \quad (C.16)$$

The sum of products of binomial coefficients in the third line of (C.15) is like a scalar product of one row of the Pascal Triangle with part of another row--the circled elements in the diagram at the right. The answer is also a member of this array (the boxed element), hence the simple result above.



Returning now to the evaluation of equation (C.5), we put (C.7) and (C.13) together with a new index $n = \frac{1}{2}(m+\mu-\nu)$. The final result is

$$M = e^{-i(\mu+\nu)\Phi} \sum_{n=-\infty}^{\infty} J_{n-\mu}(k_{\perp} r_g) J_{n+\nu}(k_{\perp} r_g) \left[\pi \delta(k_{\parallel} v_{\parallel} + n\Omega - \omega) + P \frac{1}{k_{\parallel} v_{\parallel} + n\Omega - \omega} \right], \quad (C.17)$$

which was used in (C.2) to write (3.3.15) and subsequent equations. The vanishing of (3.3.18) and (3.3.19) is shown by simultaneously changing the signs of the dummy variables \vec{k}, ω and n and using (B.4). This also accounts for the disappearance of all terms with $J_n J_{n+1}$ and of either the principle-value or the delta-function part of some other terms.

Appendix D

ALTERNATIVE FORMS FOR MOMENTUM-SPACE DIFFUSION

It is possible that the reader may consider the mixed notation of equation (3.3.22) confusing and prefer to carry through the evaluation of (3.3.14) entirely in one of the standard coordinate systems; we will record such formulae here. First, in cylindrical coordinates, the momentum-space diffusion is given by

$$\begin{aligned} \left(\frac{D\bar{F}}{Dt} \right)_{pp} &= \frac{\partial}{\partial p_{\parallel}} \left[\bar{D}_{zz} \frac{\partial \bar{F}}{\partial p_{\parallel}} \right] + \frac{1}{p_{\perp}} \frac{\partial}{\partial p_{\perp}} \left[p_{\perp} \bar{D}_{tt} \frac{\partial \bar{F}}{\partial p_{\perp}} \right] \\ &+ \frac{\partial}{\partial p_{\parallel}} \left[\bar{D}_{zt} \frac{\partial \bar{F}}{\partial p_{\perp}} \right] + \frac{1}{p_{\perp}} \frac{\partial}{\partial p_{\perp}} \left[p_{\perp} \bar{D}_{tz} \frac{\partial \bar{F}}{\partial p_{\parallel}} \right] \end{aligned} \quad (D.1)$$

with

$$\bar{D}_{zz} = \bar{\Gamma}_{zz} - \sin \theta (\bar{\Gamma}_{\theta z} + \bar{\Gamma}_{z\theta}) + \sin^2 \theta \bar{\Gamma}_{\theta\theta}, \quad (D.2)$$

$$\bar{D}_{tt} = \bar{\Gamma}_{tt} + \cos \theta (\bar{\Gamma}_{\theta t} + \bar{\Gamma}_{t\theta}) + \cos^2 \theta \bar{\Gamma}_{\theta\theta}, \quad (D.3)$$

$$\begin{aligned} \bar{D}_{zt} &= \bar{\Gamma}_{zt} + \cos \theta \bar{\Gamma}_{z\theta} - \sin \theta \bar{\Gamma}_{\theta t} - \sin \theta \cos \theta \bar{\Gamma}_{\theta\theta}. \\ tz &\quad tz \quad \theta z \quad t\theta \end{aligned} \quad (D.4)$$

The corresponding equations in spherical coordinates are

$$\begin{aligned} \left(\frac{D\bar{F}}{Dt} \right)_{pp} &= \frac{1}{p^2} \frac{\partial}{\partial p} \left[p^2 \bar{D}_{pp} \frac{\partial \bar{F}}{\partial p} \right] + \frac{1}{p^2 \sin \theta} \frac{\partial}{\partial \theta} \left[\sin \theta \bar{D}_{\theta\theta} \frac{\partial \bar{F}}{\partial \theta} \right] \\ &+ \frac{1}{p} \frac{\partial}{\partial p} \left[p \bar{D}_{p\theta} \frac{\partial \bar{F}}{\partial \theta} \right] + \frac{1}{p \sin \theta} \frac{\partial}{\partial \theta} \left[\sin \theta \bar{D}_{\theta p} \frac{\partial \bar{F}}{\partial p} \right] \end{aligned} \quad (D.5)$$

and

$$\bar{D}_{pp} = \sin^2 \theta \bar{\Gamma}_{tt} + \cos^2 \theta \bar{\Gamma}_{zz} + \sin \theta \cos \theta (\bar{\Gamma}_{tz} + \bar{\Gamma}_{zt}), \quad (D.6)$$

$$\begin{aligned}\bar{D}_{\theta\theta} = & \cos^2 \theta \bar{\Gamma}_{tt} + \sin^2 \theta \bar{\Gamma}_{zz} + \bar{\Gamma}_{\theta\theta} + \cos \theta (\bar{\Gamma}_{t\theta} + \bar{\Gamma}_{\theta t}) \\ & - \sin \theta (\bar{\Gamma}_{z\theta} + \bar{\Gamma}_{\theta z}) - \sin \theta \cos \theta (\bar{\Gamma}_{tz} + \bar{\Gamma}_{zt}) ,\end{aligned}\quad (D.7)$$

$$\begin{aligned}\bar{D}_{p\theta} = & \sin \theta \cos \theta (\bar{\Gamma}_{tt} - \bar{\Gamma}_{zz}) + \cos^2 \theta \bar{\Gamma}_{zt} - \sin^2 \theta \bar{\Gamma}_{tz} \\ & + \sin \theta \bar{\Gamma}_{t\theta} + \cos \theta \bar{\Gamma}_{z\theta} .\end{aligned}\quad (D.8)$$

Either set of equations is completely equivalent to (3.3.22) and leads to all of the same conclusions in the succeeding analysis.

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